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Abstract

This paper considers the optimal scaling problem for high-dimensional random walk Metropolis algorithms for densities which are differentiable in L^p mean but which may be irregular at some points (like the Laplace density for example) and / or are supported on an interval. Our main result is the weak convergence of the Markov chain (appropriately rescaled in time and space) to a Langevin diffusion process as the dimension d goes to infinity. Because the log-density might be non-differentiable, the limiting diffusion could be singular. The scaling limit is established under assumptions which are much weaker than the one used in the original derivation of [6]. This result has important practical implications for the use of random walk Metropolis algorithms in Bayesian frameworks based on sparsity inducing priors.

1 Introduction

A wealth of contributions have been devoted to the study of the behaviour of high-dimensional Markov chains. One of the most powerful approaches for that purpose is the scaling analysis, introduced by [6]. Assume that the target distribution has a density with respect to the d-dimensional Lebesgue measure given by:

$$\pi^d(x^d) = \prod_{i=1}^{d} \pi(x_i^d) \,. \tag{1}$$

The Random Walk Metropolis-Hastings (RWM) updating scheme was first applied in [4] and proceeds as follows. Given the current state X_k^d, a new value $Y_{k+1}^d = (Y_{k+1,i}^d)_{i=1}^d$ is obtained by moving independently each coordinate, i.e. $Y_{k+1,i}^d = X_{k,i}^d + \ell d^{-1/2} Z_{k+1}^d$ where $\ell > 0$ is a scaling factor and $(Z_k)_{k\geq 1}$ is a sequence of independent and identically distributed (i. i. d.) Gaussian random variables. Here ℓ governs the overall size of the proposed jump and plays a crucial role in determining the efficiency of the algorithm. The proposal is then accepted or rejected according to the acceptance probability $\alpha(X_k^d, Y_{k+1}^d)$ where $\alpha(x^d, y^d) = 1 \wedge \pi^d(y^d)/\pi^d(x^d)$. If the proposed value is accepted it becomes the next current value, otherwise the current value is left unchanged:

$$X_{k+1}^d = X_k^d + \ell d^{-1/2} Z_{k+1}^d \mathbb{1}_{\mathcal{A}_{k+1}^d} \,, \tag{2}$$

$$\mathcal{A}_{k+1}^d = \left\{ U_{k+1} \leq \prod_{i=1}^{d} \pi(X_{k,i}^d + \ell d^{-1/2} Z_{k+1,i}^d)/\pi(X_{k,i}^d) \right\} \,, \tag{3}$$

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where $(U_k)_{k\geq 1}$ of i. i. d. uniform random variables on $[0,1]$ independent of $(Z_k)_{k\geq 1}$.

Under certain regularity assumptions on π, it has been proved in [6] that if the X_0^d is distributed under the stationary distribution π^d, then each component of $(X_k^d)_{k\geq 0}$ appropriately rescaled in time converges weakly to a Langevin diffusion process with invariant distribution π as $d \to +\infty$.

This result allows to compute the asymptotic mean acceptance rate and to derive a practical rule to tune the factor ℓ. It is shown in [6] that the speed of the limiting diffusion has a function of ℓ has a unique maximum. The corresponding mean acceptance rate in stationarity is equal to 0.234.

These results have been derived for target distributions of the form (1) where $\pi(x) \propto \exp(-V(x))$ where V is three-times continuously differentiable. Therefore, they do not cover the cases where the target density is continuous but not smooth, for example the Laplace distribution which plays a key role as a sparsity-inducing prior in high-dimensional Bayesian inference.

The aim of this paper is to extend the scaling results for the RWM algorithm introduced in the seminal paper [6, Theorem 3] to densities which are absolutely continuous densities differentiable in L^p mean (DLM) for some $p \geq 2$ but can be either non-differentiable at some points or are supported on an interval. As shown in [3, Section 17.3], differentiability of the square root of the density in L^2 norm implies a quadratic approximation property for the log-likelihood known as local asymptotic normality. As shown below, the DLM permits the quadratic expansion of the log-likelihood without paying the twice-differentiability price usually demanded by such a Taylor expansion (such expansion of the log-likelihood plays a key role in [6]).

The paper is organised as follows. In Section 2 the target density π is assumed to be positive on $\mathbb{R}$. Theorem 2 proves that under the DLM assumption of this paper, the average acceptance rate and the expected square jump distance are the same as in [6]. Theorem 3 shows that under the same assumptions the rescaled in time Markov chain produced by the RWM algorithm converges weakly to a Langevin diffusion. We show that these results may be applied to a density of the form $\pi(x) \propto \exp(-\lambda\,|x| + U(x))$, where $\lambda \geq 0$ and U is a smooth function. In Section 3, we focus on the case where π is supported only on an open interval of $\mathbb{R}$. Under appropriate assumptions, Theorem 4 and Theorem 5 show that the same asymptotic results (limiting average acceptance rate and limiting Langevin diffusion associated with π) hold. We apply our results to Gamma and Beta distributions. The proofs are postponed to Section 4 and Section 5.

2 Positive Target density on $\mathbb{R}$

The key of the proof of our main result will be to show that the acceptance ratio and the expected square jump distance converge to a finite and non trivial limit. In the original proof of [6], the density of the product form (1) with

$$\pi(x) \propto \exp(-V(x)) \tag{4}$$

is three-times continuously differentiable and the acceptance ratio is expanded using the usual pointwise Taylor formula. More precisely, the log-ratio of the density evaluated at the proposed value and at the current state is given by $\sum_{i=1}^{d} \Delta V_i^d$ where

$$\Delta V_i^d = V\left(X_i^d\right) - V\left(X_i^d + \ell d^{-1/2} Z_i^d\right) , \tag{5}$$

and where X^d is distributed according to π^d and Z^d is a d-dimensional standard Gaussian random variable independent of X. Heuristically, the two leading terms are $\ell d^{-1/2} \sum_{i=1}^{d} \dot{V}(X_i^d) Z_i^d$

and $\ell^2 d^{-1} \sum_{i=1}^{d} \ddot{V}(X_i^d)(Z_i^d)^2/2$, where $\dot{V}$ and $\ddot{V}$ are the first and second derivatives of V, respectively. By the central limit theorem, this expression converges in distribution to a zero-mean Gaussian random variable with variance $\ell^2 I$ where

$$I = \int_{\mathbb{R}} \dot{V}^2(x)\pi(x)\mathrm{d}x\,. \tag{6}$$

Note that I is the Fisher information associated with the translation model $\theta \mapsto \pi(x+\theta)$ evaluated at $\theta = 0$. Under appropriate technical conditions, the second term converges almost surely to $-\ell^2 I/2$. Assuming that these limits exist, the acceptance ratio in the RWM algorithm converges to $\mathbb{E}[1 \wedge \exp(\mathsf{Z})]$ where Z is a Gaussian random variable with mean $-\ell^2 I/2$ and variance $\ell^2 I$; elementary computations show that $\mathbb{E}[1 \wedge \exp(\mathsf{Z})] = 2\Phi(-\ell/2\sqrt{I})$, where Φ stands for the cumulative distribution function of a standard normal distribution.

For $t \geq 0$, denote by Y_t^d the linear interpolation of the Markov chain $(X_k^d)_{k\geq 0}$ after time rescaling:

$$\begin{aligned} Y_t^d &= (\lceil d\,t \rceil - d\,t)\, X_{\lfloor d\,t \rfloor}^d + (d\,t - \lfloor d\,t \rfloor)\, X_{\lceil d\,t \rceil}^d & (7)\\ &= X_{\lfloor d\,t \rfloor}^d + (d\,t - \lfloor d\,t \rfloor)\, \ell d^{-1/2} Z_{\lceil d\,t \rceil}^d \mathbb{1}_{\mathcal{A}_{\lceil d\,t \rceil}^d}\,, & (8) \end{aligned}$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the lower and the upper integer part functions. Note that for all $k \geq 0$, $Y_{k/d}^d = X_k^d$. Denote by $(B_t, t \geq 0)$ the standard Brownian motion.

Theorem 1 ([6]). *Suppose that the target π^d and the proposal distribution are given by* (1)-(4) *and* (2) *respectively. Assume that*

(i) V is twice continuously differentiable and $\dot{V}$ is Lipshitz continuous ;

(ii) $\mathbb{E}[(\dot{V}(X))^8] < \infty$ and $\mathbb{E}[(\ddot{V}(X))^4] < \infty$ where X is distributed according to π.

Then $(Y_{t,1}^d, t \geq 0)$, where $Y_{t,1}^d$ is the first component of the vector Y_t^d defined in (7)*, converges weakly in the Wiener space (equipped with the uniform topology) to the Langevin diffusion*

$$\mathrm{d}\mathrm{Y}_t = \sqrt{h(\ell)}\mathrm{d}B_t - \frac{1}{2}h(\ell)\dot{V}(\mathrm{Y}_t)\mathrm{d}t\,, \tag{9}$$

where Y_0 is distributed according to π, $h(\ell)$ is given by

$$h(\ell) = 2\ell^2\Phi\left(-\frac{\ell}{2}\sqrt{I}\right)\,, \tag{10}$$

and I is defined in (6).

Whereas V is assumed to be twice continuously differentiable, the dual representation of the Fisher information $-\mathbb{E}[\ddot{V}(X)] = \mathbb{E}[(\dot{V}(X))^2] = I$ allows us to remove in the statement of the theorem all mention to the second derivative of V, which hints that two derivatives might not really be required. For all $\theta, x \in \mathbb{R}$, define

$$\xi_\theta(x) = \sqrt{\pi(x+\theta)}\,, \tag{11}$$

For $p \geq 1$, denote $\|f\|_{\pi,p}^p = \int |f(x)|^p \pi(x)\mathrm{d}x$. Consider the following assumptions:

H1. *There exists a measurable function $\dot{V} : \mathbb{R} \to \mathbb{R}$ such that:*

(i) There exist $p > 4$, $C > 0$ and $\beta > 1$ such that for all $\theta \in \mathbb{R}$,

$$\left\| V(\cdot + \theta) - V(\cdot) - \theta \dot{V}(\cdot) \right\|_{\pi,p} \leq C|\theta|^{\beta} .$$

(ii) The function $\dot{V}$ satisfies $\left\| \dot{V} \right\|_{\pi,6} < +\infty$.

Lemma 1. *Assume **H**1. Then, the family of densities $\theta \to \pi(\cdot + \theta)$ is Differentiable in Quadratic Mean (DQM) at $\theta = 0$ with derivative $\dot{V}$, i.e. there exists $C > 0$ such that for all $\theta \in \mathbb{R}$,*

$$\left(\int_{\mathbb{R}} \left(\xi_\theta(x) - \xi_0(x) + \theta \dot{V}(x) \xi_0(x)/2 \right)^2 \mathrm{d}x \right)^{1/2} \leq C|\theta|^{\beta} ,$$

where ξ_θ is given by (11).

Proof. The proof is postponed to Section 4.1. □

The first step in the proof is to show that the acceptance ratio $\mathbb{P}\left(\mathcal{A}_1^d\right) = \mathbb{E}(1 \wedge \exp\{\sum_{i=1}^d \Delta V_i^d\})$, and the expected square jump distance $\mathbb{E}[(Z_1^d)^2\{1 \wedge \exp(\sum_{i=1}^d \Delta V_i^d)\}]$ both converge to a finite value. To that purpose, we consider

$$\mathrm{E}^d(q) = \mathbb{E}\left[\left(Z_1^d\right)^q \left| 1 \wedge \exp\left(\sum_{i=1}^d \Delta V_i^d \right) - 1 \wedge \exp(\upsilon^d) \right| \right] ,$$

where ΔV_i^d is given by (5),

$$\upsilon^d = -\ell d^{-1/2} Z_1^d \dot{V}(X_1^d) + \sum_{i=2}^d b^d(X_i^d, Z_i^d) \tag{12}$$

$$b^d(x,z) = -\frac{\ell z}{\sqrt{d}} \dot{V}(x) + \mathbb{E}\left[2\zeta^d(X_1^d, Z_1^d) \right] - \frac{\ell^2}{4d} \dot{V}^2(x) , \tag{13}$$

$$\zeta^d(x,z) = \exp\left\{ \left(V\left(x\right) - V\left(x + \ell d^{-1/2} z\right) \right) /2 \right\} - 1 . \tag{14}$$

Proposition 1. *Assume **H**1 holds. Let X^d be a random variable distributed according to π^d and Z^d be a zero-mean standard Gaussian random variable, independent of X^d. Then, for any $q \geq 0$, $\lim_{d \to +\infty} \mathrm{E}^d(q) = 0$.*

Proof. The proof is postponed to Section 4.2. □

Proposition 1 shows that it is enough to consider υ^d to analyse the asymptotic behaviour of the acceptance ratio and the expected square jump distance as $d \to +\infty$. By the central limit theorem, the term $-\ell \sum_{i=2}^d (Z_i^d/\sqrt{d}) \dot{V}(X_i^d)$ in (12) converges in distribution to a zero-mean Gaussian random variable with variance $\ell^2 I$, where I is defined in (6). By Lemma 4 (Section 4.3), the second term, which is $d\,\mathbb{E}\left[2\zeta^d(X_1^d, Z_1^d)\right] = -d\,\mathbb{E}\left[(\zeta^d(X_1^d, Z_1^d))^2\right]$ converges to $-\ell^2 I/4$. The last term converges in probability to $-\ell^2 I/4$. Therefore, the two last terms plays a similar role in the expansion of the acceptance ratio as the second derivative of V in the regular case.

Theorem 2. *Assume **H**1 holds. Then, $\lim_{d \to +\infty} \mathbb{P}\left[\mathcal{A}_1^d\right] = a(\ell)$, where $a(\ell) = 2\Phi(-\sqrt{I}\ell/2)$.*

Proof. The proof is postponed to Section 4.3. □

The second result of this paper is that the sequence $\{(Y^d_{t,1})_{t\geq 0}, d \in \mathbb{N}^\star\}$ defined by (7) converges weakly to a Langevin diffusion. Let $(\mu_d)_{d\geq 1}$ be the sequence of distributions of $\{(Y^d_{t,1})_{t\geq 0}, d \in \mathbb{N}^\star\}$.

Proposition 2. *Assume **H**1 holds. Then, the sequence $(\mu_d)_{d\geq 1}$ is tight in $\mathbf{W}$.*

Proof. The proof is adapted from [2]; it is postponed to Section 4.4. □

By the Prohorov theorem, the tightness of $(\mu_d)_{d\geq 1}$ implies that this sequence has a weak limit point. We now prove that any limit point is the law of a solution to (9). For that purpose, we use the equivalence between the weak formulation of stochastic differential equations and martingale problems. The generator L of the Langevin diffusion (9) is given, for all $\phi \in C^2_c(\mathbb{R}, \mathbb{R})$, by

$$\mathrm{L}\phi(x) = \frac{h(\ell)}{2}\left(-\dot{V}(x)\dot{\phi}(x) + \ddot{\phi}(x)\right) , \tag{15}$$

where for $k \in \mathbb{N}$ and I an open subset of $\mathbb{R}$, $C^k_c(I, \mathbb{R})$ is the space of k-times differentiable functions with compact support, endowed with the topology of uniform convergence of all derivatives up to order k. We set $C^\infty_c(I, \mathbb{R}) = \bigcap_{k=0}^\infty C^k_c(I, \mathbb{R})$ and $\mathbf{W} = C(\mathbb{R}_+, \mathbb{R})$. The canonical process is denoted by $(W_t)_{t\geq 0}$ and $(\mathscr{B}_t)_{t\geq 0}$ is the associated filtration. For any probability measure μ on $\mathbf{W}$, the expectation with respect to μ is denoted by $\mathbb{E}^\mu$. A probability measure μ on $\mathbf{W}$ is said to solve the martingale problem associated with (9) if the pushforward of μ by W_0 is π and if for all $\phi \in C^\infty_c(\mathbb{R}, \mathbb{R})$, the process

$$\left(\phi(W_t) - \phi(W_0) - \int_0^t \mathrm{L}\phi(W_u)\mathrm{d}u\right)_{t\geq 0}$$

is a martingale with respect to μ and the filtration $(\mathscr{B}_t)_{t\geq 0}$, i.e. if for all $s, t \in \mathbb{R}_+$, $s \leq t$, $\mu-$a.s.

$$\mathbb{E}^\mu\left[\phi(W_t) - \phi(W_0) - \int_0^t \mathrm{L}\phi(W_u)\mathrm{d}u \middle| \mathscr{B}_s\right] = \phi(W_s) - \phi(W_0) - \int_0^s \mathrm{L}\phi(W_u)\mathrm{d}s\,.$$

H2. *The function $\dot{V}$ is continuous on $\mathbb{R}$ except on a Lebesgue-negligible set $\mathcal{D}_{\dot{V}}$ and is bounded on all compact sets of $\mathbb{R}$.*

If $\dot{V}$ satisfies **H**2, [7, Lemma 1.9, Theorem 20.1 Chapter 5] show that any solution to the martingale problem associated with (9) coincides with the law of a solution to the SDE (9), and conversely. Therefore, uniqueness in law of weak solutions to (9) implies uniqueness of the solution of the martingale problem.

Proposition 3. *Assume **H**2 holds. Assume also that for all $\phi \in C^\infty_c(\mathbb{R}, \mathbb{R})$, $m \in \mathbb{N}^*$, $g : \mathbb{R}^m \to \mathbb{R}$ bounded and continuous, and $0 \leq t_1 \leq \cdots \leq t_m \leq s \leq t$:*

$$\lim_{d\to+\infty} \mathbb{E}^{\mu_d}\left[\left(\phi\left(W_t\right) - \phi\left(W_s\right) - \int_s^t \mathrm{L}\phi\left(W_u\right)\mathrm{d}u\right) g\left(W_{t_1}, \ldots, W_{t_m}\right)\right] = 0\,. \tag{16}$$

Then, every limit point of the sequence of probability measures $(\mu_d)_{d\geq 1}$ on $\mathbf{W}$ is a solution to the martingale problem associated with (9).

Proof. The proof is postponed to Section 4.5. □

Theorem 3. *Assume **H**1 and **H**2 hold. Assume also that (9) has a unique weak solution. Then, $\left\{(Y^d_{t,1})_{t\geq 0}, d \in \mathbb{N}^*\right\}$ converges weakly to the solution $(Y_t)_{t\geq 0}$ of the Langevin equation defined by (9). Furthermore, $h(\ell)$ is maximized at the unique value of ℓ for which $a(\ell) = 0.234$, where a is defined in Theorem 2.*

Proof. The proof is postponed to Section 4.6. □

Example 1 (Bayesian Lasso). *We apply the results obtained above to a target density π on $\mathbb{R}$ given by $x \mapsto \mathrm{e}^{-V(x)}/\int_{\mathbb{R}} \mathrm{e}^{-V(y)}\mathrm{d}y$ where V is given by*

$$V : x \mapsto U(x) + \lambda\,|x| \;,$$

*where $\lambda \geq 0$ and U is twice continuously differentiable with bounded second derivative. Furthermore, $\int_{\mathbb{R}} |x|^6\,\mathrm{e}^{-V(x)}\mathrm{d}x < +\infty$. Define $\dot{V} : x \mapsto U'(x) + \lambda\,\mathrm{sign}(x)$, with $\mathrm{sign}(x) = -1$ if $x \leq 0$ and $\mathrm{sign}(x) = 1$ otherwise. We first check that **H**1(i) holds. Note that for all $x, y \in \mathbb{R}$,*

$$||x+y| - |x| - \mathrm{sign}(x)y| \leq 2|y|\mathbb{1}_{\mathbb{R}_+}(|y| - |x|)\,, \tag{17}$$

which implies that, for any $p \geq 1$, there exists C_p such that

$$\begin{aligned}\left\|V(\cdot+\theta) - V(\cdot) - \theta\dot{V}(\cdot)\right\|_{\pi,p} &\leq \|U(\cdot+\theta) - U(\cdot) - \theta U'(\cdot)\|_{\pi,p} + \lambda\,\||\cdot+\theta| - |\cdot| - \theta\,\mathrm{sign}(\cdot)\|_{\pi,p} \\ &\leq \|U''\|_{\infty}\,\theta^2 + 2\,|\theta|\,\lambda\{\pi([-\theta,\theta])\}^{1/p} \leq C\,|\theta|^{p+1/p} \vee |\theta|^2 \;.\end{aligned}$$

*Assumptions **H**1(ii) and **H**2 are easy to check. The uniqueness in law of (9) is established in [1, Theorem 4.5 (i)]. Therefore, Theorem 3 can be applied.*

3 Target density supported on an interval of $\mathbb{R}$

We now extend our results to densities supported by a open interval $\mathcal{I} \subset \mathbb{R}$:

$$\pi(x) \propto \exp(-V(x))\mathbb{1}_{\mathcal{I}}(x)\,,$$

where $V : \mathcal{I} \to \mathbb{R}$ is a measurable function. Note that by convention $V(x) = -\infty$ for all $x \notin \mathcal{I}$. Denote by $\overline{\mathcal{I}}$ the closure of $\mathcal{I}$ in $\mathbb{R}$. The results of Section 2 cannot be directly used in such a case, as π is no longer positive on $\mathbb{R}$. Consider the following assumption.

G1. *There exists a measurable function $\dot{V} : \mathcal{I} \to \mathbb{R}$ and $\mathrm{r} > 1$ such that:*

(i) There exist $p > 4$, $C > 0$ and $\beta > 1$ such that for all $\theta \in \mathbb{R}$,

$$\left\|\{V(\cdot+\theta) - V(\cdot)\}\mathbb{1}_{\mathcal{I}}(\cdot + \mathrm{r}\theta)\mathbb{1}_{\mathcal{I}}(\cdot + (1-\mathrm{r})\theta) - \theta\dot{V}(\cdot)\right\|_{\pi,p} \leq C|\theta|^{\beta}\,,$$

with the convention $0 \times \infty = 0$.

(ii) The function $\dot{V}$ satisfies $\|\dot{V}\|_{\pi,6} < +\infty$.

(iii) There exist $\gamma \geq 6$ and $C > 0$ such that, for all $\theta \in \mathbb{R}$,

$$\int_{\mathbb{R}} \mathbb{1}_{\mathcal{I}^c}(x+\theta)\pi(x)\mathrm{d}x \leq C|\theta|^{\gamma}\,.$$

As an important consequence of **G**1(iii), if X is distributed according to π and is independent of the standard random variable Z, there exists a constant C such that

$$\mathbb{P}\left(X + \ell d^{-1/2}Z \in \mathcal{I}^c\right) \leq C d^{-\gamma/2}\,. \tag{18}$$

Theorem 4. *Assume* **G***1 holds. Then,* $\lim_{d\to+\infty} \mathbb{P}\left[\mathcal{A}_1^d\right] = a(\ell)$*, where* $a(\ell) = 2\Phi(-\sqrt{I}\ell/2)$*.*

Proof. The proof is postponed to Section 5.1. □

We now established the weak convergence of the sequence $\{(Y_{t,1}^d)_{t\geq 0}, d \in \mathbb{N}^\star\}$, following the same steps as for the proof of Theorem 3. Denote for all $d \geq 1$, μ_d the law of the process $(Y_{t,1}^d)_{t\geq 0}$.

Proposition 4. *Assume* **G***1 holds. Then, the sequence* $(\mu_d)_{d\geq 1}$ *is tight in* $\mathbf{W}$*.*

Proof. The proof is postponed to Section 5.2. □

Contrary to the case where π is positive on $\mathbb{R}$, we do not assume that $\dot{V}$ is bounded on all compact sets of $\mathbb{R}$. Therefore, we consider the local martingale problem associated with (9): with the notations of Section 2, a probability measure μ on $\mathbf{W}$ is said to solve the local martingale problem associated with (9) if the pushforward of μ by W_0 is π and if for all $\psi \in C^\infty(\mathbb{R},\mathbb{R})$, the process

$$\left(\psi(W_t) - \psi(W_0) - \int_0^t \mathrm{L}\psi(W_u)\mathrm{d}u\right)_{t\geq 0}$$

is a local martingale with respect to μ and the filtration $(\mathscr{B}_t)_{t\geq 0}$. By [1, Theorem 1.27], any solution to the local martingale problem associated with (9) coincides with the law of a solution to the SDE (9) and conversely. If (9) admits a unique solution in law, this law is the unique solution to the local martingale problem associated with (9). We first prove that any limit point μ of $(\mu_d)_{d\geq 1}$ is a solution to the local martingale problem associated with (9).

G2. *The function* $\dot{V}$ *is continuous on* $\mathcal{I}$ *except on a null-set* $\mathcal{D}_{\dot{V}}$*, with respect to the Lebesgue measure, and is bounded on all compact sets of* $\mathcal{I}$*.*

This condition does not preclude that $\dot{V}$ is unbounded at the boundary of $\mathcal{I}$.

Proposition 5. *Assume* **G***1 and* **G***2 hold. Assume also that for all* $\phi \in C_c^\infty(\mathcal{I},\mathbb{R})$*,* $m \in \mathbb{N}^*$*,* $g : \mathbb{R}^m \to \mathbb{R}$ *bounded and continuous, and* $0 \leq t_1 \leq \cdots \leq t_m \leq s \leq t$*:*

$$\lim_{d\to+\infty} \mathbb{E}^{\mu_d}\left[\left(\phi\left(W_t\right) - \phi\left(W_s\right) - \int_s^t \mathrm{L}\phi\left(W_u\right)\mathrm{d}u\right) g\left(W_{t_1},\ldots,W_{t_m}\right)\right] = 0\,. \tag{19}$$

Then, every limit point of the sequence of probability measures $(\mu_d)_{d\geq 1}$ *on* $\mathbf{W}$ *is a solution to the local martingale problem associated with* (9)*.*

Proof. The proof is postponed to Section 5.3. □

Theorem 5. *Assume* **G***1 and* **G***2 hold. Assume also that* (9) *has a unique weak solution. Then,* $\left\{(Y_{t,1}^d)_{t\geq 0}, d \in \mathbb{N}^*\right\}$ *converges weakly to the solution* $(Y_t)_{t\geq 0}$ *of the Langevin equation defined by* (9)*. Furthermore,* $h(\ell)$ *is maximized at the unique value of* ℓ *for which* $a(\ell) = 0.234$*, where* a *is defined in Theorem 2.*

Proof. The proof is along the same lines as the proof of Theorem 3 and is postponed to Section 5.4. □

The conditions for uniqueness in law of singular one-dimensional stochastic differential equations are given in [1]. These conditions are rather involved and difficult to summarize in full generality. We rather illustrate Theorem 5 by two examples.

Example 2 (Application to the Gamma distribution)**.** *Define the class of the generalized Gamma distributions as the family of densities on* $\mathbb{R}$ *given by*

$$\pi_\gamma : x \mapsto x^{\mathrm{a}_1 - 1} \exp(-x^{\mathrm{a}_2}) \mathbb{1}_{\mathbb{R}_+^\star}(x) \Big/ \int_{\mathbb{R}_+^\star} y^{\mathrm{a}_1-1} \exp(-y^{\mathrm{a}_2}) \mathrm{d}y \,,$$

with two parameters $\mathrm{a}_1 > 6$ *and* $\mathrm{a}_2 > 0$*. Note that in this case* $\mathcal{I} = \mathbb{R}_+^\star$*, for all* $x \in \mathcal{I}$*,* $V_\gamma : x \mapsto x^{\mathrm{a}_2} - (\mathrm{a}_1 - 1)\log x$ *and* $\dot{V}_\gamma : x \mapsto \mathrm{a}_2 x^{\mathrm{a}_2 - 1} - (\mathrm{a}_1 - 1)/x$*. We check that* ***G****1 holds with* $\mathrm{r} = 3/2$*. First, we show that* ***G****1(i) holds with* $p = 5$*. Write for all* $\theta \in \mathbb{R}$ *and* $x \in \mathcal{I}$*,*

$$\{V_\gamma(x+\theta) - V_\gamma(x)\} \mathbb{1}_{\mathcal{I}}(x + (1-r)\theta) \mathbb{1}_{\mathcal{I}}(x + r\theta) - \theta \dot{V}_\gamma(x) = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 \,,$$

where

$$\begin{aligned}
\mathcal{E}_1 &= \theta \dot{V}_\gamma(x) \{\mathbb{1}_{\mathcal{I}}(x - \theta/2)\mathbb{1}_{\mathcal{I}}(x + 3\theta/2) - 1\} \,, \\
\mathcal{E}_2 &= (1 - \mathrm{a}_1)\{\log(1 + \theta/x) - \theta/x\} \mathbb{1}_{\mathcal{I}}(x - \theta/2)\mathbb{1}_{\mathcal{I}}(x + 3\theta/2) \,, \\
\mathcal{E}_3 &= ((x+\theta)^{\mathrm{a}_2} - x^{\mathrm{a}_2} - \mathrm{a}_2 \theta x^{\mathrm{a}_2 - 1}) \mathbb{1}_{\mathcal{I}}(x - \theta/2)\mathbb{1}_{\mathcal{I}}(x + 3\theta/2) \,.
\end{aligned}$$

It is enough to prove that there exists $q > 5$ *such that for all* $i \in \{1, 2, 3\}$*,* $\int_{\mathcal{I}} |\mathcal{E}_i|^5 \, \pi_\gamma(x)\mathrm{d}x \leq C|\theta|^q$*. The result is proved for* $\theta < 0$ *(the proof for* $\theta > 0$ *follows the same lines). For all* $\theta \in \mathbb{R}$ *using* $\mathrm{a}_1 > 6$*,*

$$\begin{aligned}
\int_{\mathbb{R}_+^*} |\mathcal{E}_1|^5 \, \pi_\gamma(x)\mathrm{d}x &\leq C|\theta|^5 \int_0^{3|\theta|/2} \left\{ 1/x^5 + x^{5(\mathrm{a}_2 - 1)} \right\} x^{\mathrm{a}_1 - 1} \mathrm{e}^{-x^{\mathrm{a}_2}} \mathrm{d}x \,, \\
&\leq C|\theta|^{\mathrm{a}_1} \left(\int_0^{3/2} x^{\mathrm{a}_1 - 6} \mathrm{e}^{-(|\theta| x)^{\mathrm{a}_2}} \mathrm{d}x + |\theta|^{5\mathrm{a}_2} \int_0^{3/2} x^{5(\mathrm{a}_2 - 1) + \mathrm{a}_1 - 1} \mathrm{e}^{-(|\theta| x)^{\mathrm{a}_2}} \mathrm{d}x \right) \,, \\
&\leq C(|\theta|^{\mathrm{a}_1} + |\theta|^{5\mathrm{a}_2 + \mathrm{a}_1}) \,.
\end{aligned} \tag{20}$$

On the other hand, as for all $x > -1$*,* $x/(x+1) \leq \log(1+x) \leq x$*, for all* $\theta < 0$*, and* $x \geq 3|\theta|/2$*,*

$$|\log(1 + \theta/x) - \theta/x| \leq \frac{|\theta|^2}{x^2(1 + \theta/x)} \leq 3|\theta|^2/x^2 \,,$$

where the last inequality come from $|\theta|/x \leq 2/3$*. Then, it yields*

$$\begin{aligned}
\int_{\mathbb{R}_+^*} |\mathcal{E}_2(x)|^5 \, \pi_\gamma(x)\mathrm{d}x &\leq C|\theta|^{10} \left(\int_{3|\theta|/2}^1 x^{\mathrm{a}_1 - 11} \mathrm{e}^{-x^{\mathrm{a}_2}} \mathrm{d}x + \int_1^{+\infty} x^{\mathrm{a}_1 - 11} \mathrm{e}^{-x^{\mathrm{a}_2}} \mathrm{d}x \right) \,, \\
&\leq C(|\theta|^{\mathrm{a}_1} + |\theta|^{10}) \,.
\end{aligned} \tag{21}$$

For the last term, for all $\theta < 0$ *and all* $x \geq 3|\theta|/2$*, using a Taylor expansion of* $x \mapsto x^{\mathrm{a}_2}$*, there exists* $\zeta \in [x + \theta, x]$ *such that*

$$\left|(x+\theta)^{\mathrm{a}_2} - x^{\mathrm{a}_2} - \mathrm{a}_2 \theta x^{\mathrm{a}_2 - 1}\right| \leq C|\theta|^2 |\zeta|^{\mathrm{a}_2 - 2} \leq C|\theta|^2 \, |x|^{\mathrm{a}_2 - 2} \,.$$

Then,

$$\int_{\mathbb{R}_+^*} |\mathcal{E}_3(x)|^5 \, \pi_\gamma(x)\mathrm{d}x \leq C|\theta|^{10} \int_{3|\theta|/2}^{+\infty} x^{5(\mathrm{a}_2 - 2) + \mathrm{a}_1 - 1} \mathrm{e}^{-x^{\mathrm{a}_2}} \mathrm{d}x \leq C(|\theta|^{5\mathrm{a}_2 + \mathrm{a}_1} + |\theta|^{10}) \,. \tag{22}$$

Combining (20), (21),(22) *and using that* $\mathrm{a}_1 > 6$ *concludes the proof of* ***G****1(i) for* $p = 5$. *The proof of* ***G****1(ii) follows from*

$$\int_{\mathbb{R}_+^*} |\dot{V}_\gamma(x)|^6 \pi_\gamma(x)\mathrm{d}x \le C\left(\int_{\mathbb{R}_+^*} x^{\mathrm{a}_1-1+6(\mathrm{a}_2-1)}\mathrm{e}^{-x^{\mathrm{a}_2}}\mathrm{d}x + \int_{\mathbb{R}_+^*} x^{\mathrm{a}_1-7}\mathrm{e}^{-x^{\mathrm{a}_2}}\mathrm{d}x\right) < \infty$$

and ***G****1(iii) follows from* $\int_{\mathbb{R}} \mathbb{1}_{\mathcal{I}^c}(x+\theta)\pi_\gamma(x)\mathrm{d}x \le C|\theta|^{\mathrm{a}_1}$. *Now consider the Langevin equation associated with* π_γ *given by* $\mathrm{d}Y_t = -\dot{V}_\gamma(Y_t)\mathrm{d}t + \sqrt{2}\mathrm{d}B_t$ *with initial distribution* π_γ. *This stochastic differential equation has* 0 *as singular point, which has right type* 3 *according to the terminology of [1]. On the other hand* ∞ *has type A and the existence and uniqueness in law for the SDE follows from [1, Theorem 4.6 (viii)]. Since* ***G****2 is straightforward, Theorem 5 can be applied.*

Example 3 (Application to the Beta distribution)**.** *Consider now the case of the Beta distributions* π_β *with density* $x \mapsto x^{a_1-1}(1-x)^{a_2-1}\mathbb{1}_{(0,1)}(x)$ *with* $a_1, a_2 > 6$. *Here* $\mathcal{I} = (0,1)$ *and the log-density* V_β *and its derivative on* $\mathcal{I}$ *are defined by* $V_\beta(x) = -(a_1-1)\log x - (a_2-1)\log(1-x)$ *and* $\dot{V}_\beta(x) = -(a_1-1)/x - (a_2-1)/(1-x)$. *Along the same lines as above,* π_β *satisfies* ***G****1 and* ***G****2. Hence Theorem 4 can be applied if we establish the uniqueness in law for the Langevin equation associated with* π_β *defined by* $\mathrm{d}Y_t = -\dot{V}_\beta(Y_t)\mathrm{d}t + \sqrt{2}\mathrm{d}B_t$ *with initial distribution* π_β. *In the terminology of [1],* 0 *has right type* 3 *and* 1 *has left type* 3. *Therefore by [1, Theorem 2.16 (i), (ii)], the SDE has a global unique weak solution. To illustrate our findings, consider the Beta distribution with parameters* $a_1 = 10$ *and* $a_2 = 10$. *Define the expected square distance by* $\mathrm{ESJD}^d(\ell) = \mathbb{E}\left[\left\|X_1^d - X_0^d\right\|^2\right]$ *where* X_0^d *has distribution* π_β^d *and* X_1^d *is the first iterate of the Markov chain defined by the Random Walk Metropolis algorithm given in* (2). *By Theorem 4 and Theorem 5, we have* $\lim_{d\to+\infty}\mathrm{ESJD}^d(\ell) = h(\ell) = \ell^2 a(\ell)$. *Figure 1 displays an empirical estimation for the* ESJD^d *for dimensions* $d = 10, 50, 100$ *as a function of the empirical mean acceptance rate. We can observe that as expected, the* ESJD^d *converges to some limit function as d goes infinity, and this function has a maximum for a mean acceptance probability around* 0.23.

4 Proofs of Section 2

For any real random variable Y and any $p \ge 1$, let $\|Y\|_p := \mathbb{E}[|Y|^p]^{1/p}$.

4.1 Proof of Lemma 1

Let $\Delta_\theta V(x) = V(x) - V(x+\theta)$. By definition of ξ_θ and π,

$$\left(\xi_\theta(x) - \xi_0(x) + \theta\dot{V}(x)\xi_0(x)/2\right)^2 \le 2\left\{A_\theta(x) + B_\theta(x)\right\}\pi(x)\,,$$

where

$$\begin{aligned} A_\theta(x) &= \left(\exp(\Delta_\theta V(x)/2) - 1 - \Delta_\theta V(x)/2\right)^2\,, \\ B_\theta(x) &= \left(\Delta_\theta V(x) + \theta\dot{V}(x)\right)^2/4. \end{aligned}$$

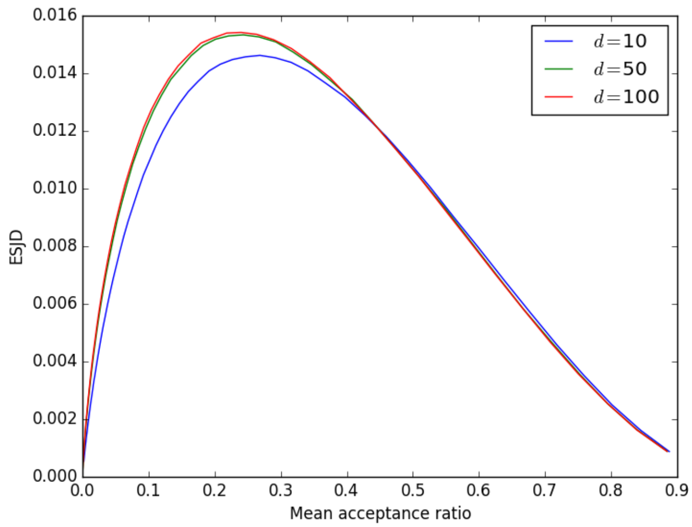

Figure 1: Expected square jumped distance for the beta distribution with parameters $a_1 = 10$ and $a_2 = 10$ as a function of the mean acceptance rate for $d = 10, 50, 100$.

By **H**1(i), $\|B_\theta\|_{\pi,p} \leq C|\theta|^\beta$. For A_θ, note that for all $x \in \mathbb{R}$, $(\exp(x)-1-x)^2 \leq 2x^4(\exp(2x)+1)$. Then,

$$\begin{aligned}\int_{\mathbb{R}} A_\theta(x)\pi(x)\mathrm{d}x &\leq C\int_{\mathbb{R}} \Delta_\theta V(x)^4\left(1+\mathrm{e}^{\Delta_\theta V(x)}\right)\pi(x)\mathrm{d}x \\ &\leq C\int_{\mathbb{R}}\left(\Delta_\theta V(x)^4 + \Delta_{-\theta}V(x)^4\right)\pi(x)\mathrm{d}x\,.\end{aligned}$$

The proof is completed writing (the same inequality holds for $\Delta_{-\theta}V$):

$$\int_{\mathbb{R}} \Delta_\theta V(x)^4\pi(x)\mathrm{d}x \leq C\left[\int_{\mathbb{R}}\left(\Delta_\theta V(x) - \theta\dot{V}(x)\right)^4\pi(x)\mathrm{d}x + \theta^4\int_{\mathbb{R}}\dot{V}^4(x)\pi(x)\mathrm{d}x\right]$$

and using **H**1(i)-(ii).

4.2 Proof of Proposition 1

Define

$$R(x) = \int_0^x \frac{(x-u)^2}{(1+u)^3}\mathrm{d}u\,. \tag{23}$$

R is the remainder term of the Taylor expansion of $x \mapsto \log(1+x)$:

$$\log(1+x) = x - x^2/2 + R(x)\,. \tag{24}$$

We preface the proof by the following Lemma.

Lemma 2. *Assume **H**1 holds. Then, if X is a random variable distributed according to π and Z is a standard Gaussian random variable independent of X,* *(i)* $\lim_{d\to+\infty} d\ \left\|\zeta^d(X,Z)+\ell Z\dot{V}(X)/(2\sqrt{d})\right\|_2^2=0.$

(ii) $\lim_{d\to+\infty}\sqrt{d}\left\|V(X)-V(X+\ell Z/\sqrt{d})+\ell Z\dot{V}(X)/\sqrt{d}\right\|_p=0.$

(iii) $\lim_{d\to\infty} d\left\|R\left(\zeta^d(X,Z)\right)\right\|_1=0,$

where ζ^d is given by (14).

Proof. Using the definitions (11) and (14) of ζ^d and ξ_θ ,

$$\zeta^d(x,z)=\xi_{\ell z d^{-1/2}}(x)/\xi_0(x)-1\,. \tag{25}$$

(i) The proof follows from Lemma 1 using that $\beta>1$:

$$\left\|\zeta^d(X,Z)+\ell Z\dot{V}(X)/(2\sqrt{d})\right\|_2^2\leq C\ell^{2\beta}d^{-\beta}\mathbb{E}\left[|Z|^{2\beta}\right]\,.$$

(ii) Using **H**1(i), we get that

$$\left\|V(X)-V(X+\ell Z/\sqrt{d})+\ell Z\dot{V}(X)/\sqrt{d}\right\|_p^p\leq C\ell^{\beta p}d^{-\beta p/2}\mathbb{E}\left[|Z|^{\beta p}\right]$$

and the proof follows since $\beta>1$.

(iii) Note that for all $x>0$, $u\in[0,x]$, $|(x-u)(1+u)^{-1}|\leq|x|$, and the same inequality holds for $x\in(-1,0]$ and $u\in[x,0]$. Then by (23) and (24), for all $x>-1$, $|R(x)|\leq x^2\left|\log(1+x)\right|$. Then by (50), setting $\Psi_d(x,z)=R(\zeta^d(x,z))$

$$|\Psi_d(x,z)|\leq\left(\xi_{\ell z d^{-1/2}}(x)/\xi_0(x)-1\right)^2\left|V(x+\ell z d^{-1/2})-V(x)\right|/2\,.$$

Since for all $x\in\mathbb{R}$, $|\exp(x)-1|\leq|x|(\exp(x)+1)$, this yields,

$$|\Psi_d(x,z)|\leq 4^{-1}\left|V(x+\ell z d^{-1/2})-V(x)\right|^3\left(\exp\left(V(x)-V(x+\ell z d^{-1/2})\right)+1\right)\,,$$

which implies that

$$\int_{\mathbb{R}}|\Psi_d(x,z)|\,\pi(x)\mathrm{d}x\leq 4^{-1}\int_{\mathbb{R}}\left|V(x+\ell z d^{-1/2})-V(x)\right|^3\{\pi(x)+\pi(x+\ell z d^{-1/2})\}\mathrm{d}x\,.$$

By Hölder's inequality and using **H**1(i),

$$\int_{\mathbb{R}}|\Psi_d(x,z)|\,\pi(x)\mathrm{d}x\leq C\left(\left|\ell z d^{-1/2}\right|^3\left(\int_{\mathbb{R}}\left|\dot{V}(x)\right|^4\pi(x)\mathrm{d}x\right)^{3/4}+\left|\ell z d^{-1/2}\right|^{3\beta}\right)\,.$$

The proof follows from **H**1(ii) since $\beta>1$.

□

For all $d\geq1$, let X^d be distributed according to π^d, and Z^d be d-dimensional Gaussian random variable independent of X^d, set

$$\mathrm{J}^d=\left\|\sum_{i=2}^d\left\{\Delta V_i^d-b^d(X_i^d,Z_i^d)\right\}\right\|_1\,,$$

where ΔV_i^d and b^d are defined in (5) and (13), respectively.

Lemma 3. $\lim_{d\to+\infty} \mathrm{J}^d = 0$.

Proof. Noting that $\Delta V_i^d = 2\log\left(1+\zeta^d\left(X_i^d, Z_i^d\right)\right)$ and using (24), we get

$$\mathrm{J}^d \le \sum_{i=1}^{3} \mathrm{J}_i^d = \left\| \sum_{i=2}^{d} 2\zeta^d\left(X_i^d, Z_i^d\right) + \frac{\ell Z_i^d}{\sqrt{d}} \dot{V}(X_i^d) - \mathbb{E}\left[2\zeta^d(X_i^d, Z_i^d)\right] \right\|_1 + \left\| \sum_{i=2}^{d} \zeta^d\left(X_i^d, Z_i^d\right)^2 - \frac{\ell^2}{4d}\dot{V}^2(X_i^d) \right\|_1 + 2\left\| \sum_{i=2}^{d} R\left(\zeta^d\left(X_i^d, Z_i^d\right)\right) \right\|_1 ,$$

where R is defined by (23). By Lemma 2(i), the first term goes to 0 as d goes to $+\infty$ since

$$\mathrm{J}_1^d \le \sqrt{d}\left\| 2\zeta^d\left(X_1^d, Z_1^d\right) + \frac{\ell Z_1^d}{\sqrt{d}}\dot{V}(X_1^d) \right\|_2 .$$

Consider now J_2^d. We use the following decomposition for all $2 \le i \le d$,

$$\zeta^d(X_i^d, Z_i^d)^2 - \frac{\ell^2}{4d}\dot{V}^2(X_i^d) = \left(\zeta^d(X_i^d, Z_i^d) + \frac{\ell}{2\sqrt{d}} Z_i^d \dot{V}(X_i^d)\right)^2 - \frac{\ell}{\sqrt{d}} Z_i^d \dot{V}(X_i^d)\left(\zeta^d(X_i^d, Z_i^d) + \frac{\ell}{2\sqrt{d}} Z_i^d \dot{V}(X_i^d)\right) + \frac{\ell^2}{4d}\left\{(Z_i^d)^2 - 1\right\}\dot{V}^2(X_i^d) .$$

Then,

$$\mathrm{J}_2^d \le d\left\| \zeta^d(X_1^d, Z_1^d) + \frac{\ell}{2\sqrt{d}} Z_1^d \dot{V}(X_1^d) \right\|_2^2 + \frac{\ell^2}{4d}\left\| \sum_{i=2}^{d} \dot{V}^2(X_i^d)\left\{(Z_i^d)^2 - 1\right\} \right\|_1 + \ell\sqrt{d}\left\| \dot{V}(X_1^d) Z_1^d \left(\zeta^d(X_1^d, Z_1^d) + \frac{\ell}{2\sqrt{d}} Z_1^d \dot{V}(X_1^d)\right) \right\|_1 .$$

Using **H**1(ii), Lemma 2(i) and the Cauchy-Schwarz inequality show that the first and the last term converge to zero. For the second term note that $\mathbb{E}\left[(Z_i^d)^2 - 1\right] = 0$ so that

$$d^{-1}\left\| \sum_{i=2}^{d} \dot{V}^2(X_i^d)\left\{(Z_i^d)^2 - 1\right\} \right\|_1 \le d^{-1/2}\,\mathrm{Var}\left[\dot{V}^2(X_1^d)\left\{(Z_1^d)^2 - 1\right\}\right]^{1/2} \to 0 .$$

Finally, $\lim_{d\to\infty} \mathrm{J}_3^d = 0$ by (24) and Lemma 2(iii). $\square$

Proof of Proposition 1. Let $q > 0$ and $\Lambda^d = -\ell d^{-1/2} Z_1^d \dot{V}(X_1^d) + \sum_{i=2}^{d} \Delta V_i^d$. By the triangle inequality, $\mathrm{E}^d(q) \le \mathrm{E}_1^d(q) + \mathrm{E}_2^d(q)$ where

$$\mathrm{E}_1^d(q) = \mathbb{E}\left[\left(Z_1^d\right)^q \left| 1 \wedge \exp\left\{\sum_{i=1}^{d} \Delta V_i^d\right\} - 1 \wedge \exp\left\{\Lambda^d\right\} \right|\right] ,$$

$$\mathrm{E}_2^d(q) = \mathbb{E}\left[\left(Z_1^d\right)^q \left| 1 \wedge \exp\left\{\Lambda^d\right\} - 1 \wedge \exp\left\{\upsilon^d\right\} \right|\right] .$$

Since $t \mapsto 1 \wedge \mathrm{e}^t$ is 1-Lipschitz, by the Cauchy-Schwarz inequality we get

$$\mathrm{E}_1^d(q) \le \left\| Z_1^d \right\|_{2q}^q \left\| \Delta V_1^d + \ell d^{-1/2} Z_1^d \dot{V}(X_1^d) \right\|_2 .$$

By Lemma 2(ii), $\mathrm{E}_1^d(q)$ goes to 0 as d goes to $+\infty$. Consider now $\mathrm{E}_2^d(q)$. Using again that $t \mapsto 1 \wedge \mathrm{e}^t$ is 1-Lipschitz and Lemma 3, $\mathrm{E}_2^d(q)$ goes to 0. $\square$

4.3 Proof of Theorem 2

Following [2], we introduce the function $\mathcal{G}$ defined on $\overline{\mathbb{R}}_+ \times \mathbb{R}$ by:

$$\mathcal{G}(a,b) = \begin{cases} \exp\left(\frac{a-b}{2}\right)\Phi\left(\frac{b}{2\sqrt{a}} - \sqrt{a}\right) & \text{if } a \in (0,+\infty)\,, \\ 0 & \text{if } a = +\infty\,, \\ \exp\left(-\frac{b}{2}\right)\mathbb{1}_{\{b>0\}} & \text{if } a = 0\,, \end{cases} \tag{26}$$

where Φ is the cumulative distribution function of a standard normal variable, and Γ:

$$\Gamma(a,b) = \begin{cases} \Phi\left(-\frac{b}{2\sqrt{a}}\right) + \exp\left(\frac{a-b}{2}\right)\Phi\left(\frac{b}{2\sqrt{a}} - \sqrt{a}\right) & \text{if } a \in (0,+\infty)\,, \\ \frac{1}{2} & \text{if } a = +\infty\,, \\ \exp\left(-\frac{b_+}{2}\right) & \text{if } a = 0\,. \end{cases} \tag{27}$$

Note that $\mathcal{G}$ and Γ are bounded on $\overline{\mathbb{R}}_+ \times \mathbb{R}$. $\mathcal{G}$ and Γ are used throughout Section 4.

Lemma 4. *Assume **H**1 holds. For all $d \in \mathbb{N}^*$, let X^d be a random variable distributed according to π^d and Z^d be a standard Gaussian random variable in $\mathbb{R}^d$, independent of X. Then,*

$$\lim_{d\to+\infty} d\,\mathbb{E}\left[2\zeta^d(X_1^d, Z_1^d)\right] = -\frac{\ell^2}{4} I\,,$$

where I is defined in (6) *and ζ^d in* (14).

Proof. By (14),

$$\begin{aligned} d\,\mathbb{E}\left[2\zeta^d(X_1^d, Z_1^d)\right] &= 2d\mathbb{E}\left[\int_{\mathbb{R}} \sqrt{\pi\left(x + \ell d^{-1/2} Z_1^d\right)}\sqrt{\pi\left(x\right)}\mathrm{d}x - 1\right]\,, \\ &= -d\mathbb{E}\left[\int_{\mathbb{R}}\left(\sqrt{\pi\left(x + \ell d^{-1/2}\, Z_1^d\right)} - \sqrt{\pi\left(x\right)}\right)^2 \mathrm{d}x\right] = -d\,\mathbb{E}\left[\{\zeta^d(X_1^d, Z_1^d)\}^2\right]\,. \end{aligned}$$

The proof is then completed by Lemma 2(i). □

Proof of Theorem 2. By definition of $\mathcal{A}_1^d$, see (3),

$$\mathbb{P}\left[\mathcal{A}_1^d\right] = \mathbb{E}\left[1 \wedge \exp\left\{\sum_{i=1}^d \Delta V_i^d\right\}\right]\,,$$

where $\Delta V_i^d = V(X_{0,i}^d) - V(X_{0,i}^d + \ell d^{-1/2} Z_{1,i}^d)$ and where X_0^d is distributed according to π^d and independent of the standard d-dimensional Gaussian random variable Z_1^d. Following the same steps as in the proof of Proposition 1 yields:

$$\lim_{d\to+\infty} \left|\mathbb{P}\left[\mathcal{A}_1^d\right] - \mathbb{E}\left[1 \wedge \exp\left\{\Theta^d\right\}\right]\right| = 0\,, \tag{28}$$

where

$$\Theta^d = -\ell d^{-1/2}\sum_{i=1}^d Z_{1,i}^d \dot{V}(X_{0,i}^d) - \ell^2 \sum_{i=2}^d \dot{V}(X_{0,i}^d)^2/(4d) + 2(d-1)\mathbb{E}\left[\zeta^d(X_{0,1}^d, Z_{1,1}^d)\right]\,.$$

Conditional on X_0^d, Θ^d is a one dimensional Gaussian random variable with mean μ_d and variance σ_d^2, defined by

$$\mu_d = -\ell^2 \sum_{i=2}^{d} \dot{V}(X_{0,i}^d)^2/(4d) + 2(d-1)\mathbb{E}\left[\zeta^d(X_{0,1}^d, Z_{1,1}^d)\right]$$

$$\sigma_d^2 = \ell^2 d^{-1} \sum_{i=1}^{d} \dot{V}(X_{0,i}^d)^2 \,.$$

Therefore, since for any $G \sim \mathcal{N}(\mu, \sigma^2)$, $\mathbb{E}[1 \wedge \exp(G)] = \Phi(\mu/\sigma) + \exp(\mu + \sigma^2/2)\Phi(-\sigma - \mu/\sigma)$, taking the expectation conditional on X_0^d, we have

$$\begin{aligned}\mathbb{E}\left[1 \wedge \exp\left\{\Theta^d\right\}\right] &= \mathbb{E}\left[\Phi(\mu_d/\sigma_d) + \exp(\mu_d + \sigma_d^2/2)\Phi(-\sigma_d - \mu_d/\sigma_d)\right] \\ &= \mathbb{E}\left[\Gamma(\sigma_d^2, -2\mu_d)\right] \,,\end{aligned}$$

where the function Γ is defined in (27). By Lemma 4 and the law of large numbers, almost surely, $\lim_{d\to+\infty} \mu_d = -\ell^2 I/2$ and $\lim_{d\to+\infty} \sigma_d^2 = \ell^2 I$. Thus, as Γ is bounded, by Lebesgue's dominated convergence theorem:

$$\lim_{d\to+\infty} \mathbb{E}\left[1 \wedge \exp\left\{\Theta^d\right\}\right] = 2\Phi\left(-\ell\sqrt{I}/2\right) \,.$$

The proof is then completed by (28). □

4.4 Proof of Proposition 2

By Kolmogorov's criterion it is enough to prove that there exists a non-decreasing function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $d \geq 1$ and all $0 \leq s \leq t$,

$$\mathbb{E}\left[\left(Y_{t,1}^d - Y_{s,1}^d\right)^4\right] \leq \gamma(t)(t-s)^2 \,.$$

The inequality is straightforward for all $0 \leq s \leq t$ such that $\lfloor ds \rfloor = \lfloor dt \rfloor$. For all $0 \leq s \leq t$ such that $\lceil ds \rceil \leq \lfloor dt \rfloor$,

$$Y_{t,1}^d - Y_{s,1}^d = X_{\lfloor dt \rfloor,1}^d - X_{\lceil ds \rceil,1}^d + \frac{dt - \lfloor dt \rfloor}{\sqrt{d}} \ell Z_{\lceil dt \rceil,1}^d \mathbb{1}_{\mathcal{A}_{\lceil dt \rceil}^d} + \frac{\lceil ds \rceil - ds}{\sqrt{d}} \ell Z_{\lceil ds \rceil,1}^d \mathbb{1}_{\mathcal{A}_{\lceil ds \rceil}^d} \,.$$

Then by the Hölder inequality,

$$\mathbb{E}\left[\left(Y_{t,1}^d - Y_{s,1}^d\right)^4\right] \leq C\left((t-s)^2 + \mathbb{E}\left[\left(X_{\lfloor dt \rfloor,1}^d - X_{\lceil ds \rceil,1}^d\right)^4\right]\right) \,,$$

where we have used

$$\frac{(dt - \lfloor dt \rfloor)^2}{d^2} + \frac{(\lceil ds \rceil - ds)^2}{d^2} \leq \frac{(dt-ds)^2 + (\lceil ds \rceil - \lfloor dt \rfloor)^2}{d^2} \leq 2(t-s)^2 \,.$$

The proof is completed using Lemma 5.

Lemma 5. *Assume **H1**. Then, there exists $C > 0$ such that, for all $0 \leq k_1 < k_2$,*

$$\mathbb{E}\left[\left(X_{k_2,1}^d - X_{k_1,1}^d\right)^4\right] \leq C \sum_{p=2}^{4} \frac{(k_2 - k_1)^p}{d^p} \,.$$

Proof. For all $0 \leq k_1 < k_2$,

$$\mathbb{E}\left[\left(X^d_{k_2,1} - X^d_{k_1,1}\right)^4\right] = \frac{\ell^4}{d^2}\mathbb{E}\left[\left(\sum_{k=k_1+1}^{k_2} Z^d_{k,1} - \sum_{k=k_1+1}^{k_2} Z^d_{k,1}\mathbb{1}_{\left(\mathcal{A}^d_k\right)^c}\right)^4\right] .$$

Therefore by the Hölder inequality,

$$\mathbb{E}\left[\left(X^d_{k_2,1} - X^d_{k_1,1}\right)^4\right] \leq \frac{24\ell^4}{d^2}(k_2-k_1)^2 + \frac{8\ell^4}{d^2}\mathbb{E}\left[\left(\sum_{k=k_1+1}^{k_2} Z^d_{k,1}\mathbb{1}_{\left(\mathcal{A}^d_k\right)^c}\right)^4\right] . \tag{29}$$

The second term can be written:

$$\mathbb{E}\left[\left(\sum_{k=k_1+1}^{k_2} Z^d_{k,1}\mathbb{1}_{\left(\mathcal{A}^d_k\right)^c}\right)^4\right] = \sum \mathbb{E}\left[\prod_{i=1}^{4} Z^d_{m_i,1}\mathbb{1}_{\left(\mathcal{A}^d_{m_i}\right)^c}\right] ,$$

where the sum is over all the quadruplets $(m_i)_{i=1}^4$ satisfying $m_i \in \{k_1+1,\ldots,k_2\}$, $i = 1,\ldots,4$. The expectation on the right hand side can be upper bounded depending on the cardinality of $\{m_1,\ldots,m_4\}$. For all $1 \leq j \leq 4$, define

$$\mathcal{I}_j = \{(m_1,\ldots,m_4) \in \{k_1+1,\ldots,k_2\} \ ; \ \#\{m_1,\ldots,m_4\} = j\} . \tag{30}$$

Let $(m_1,m_2,m_3,m_4) \in \{k_1+1,\ldots,k_2\}^4$ and $(\tilde{X}^d_k)_{k\geq 0}$ be defined as:

$$\tilde{X}^d_0 = X^d_0 \quad \text{and} \quad \tilde{X}^d_{k+1} = \tilde{X}^d_k + \mathbb{1}_{k\notin\{m_1-1,m_2-1,m_3-1,m_4-1\}}\frac{\ell}{\sqrt{d}}Z^d_{k+1}\mathbb{1}_{\tilde{\mathcal{A}}^d_{k+1}} ,$$

with $\tilde{\mathcal{A}}^d_{k+1} = \left\{U_{k+1} \leq \exp\left(\sum_{i=1}^d \Delta\tilde{V}^d_{k,i}\right)\right\}$, where for all $k \geq 0$ and all $1 \leq i \leq d$, $\Delta\tilde{V}_{k,i}$ is defined by

$$\Delta\tilde{V}^d_{k,i} = V\left(\tilde{X}^d_{k,i}\right) - V\left(\tilde{X}^d_{k,i} + \frac{\ell}{\sqrt{d}}Z^d_{k+1,i}\right) .$$

Note that on the event $\bigcap_{j=1}^4 \left\{\mathcal{A}^d_{m_j}\right\}^c$, the two processes $(X_k)_{k\geq 0}$ and $(\tilde{X}_k)_{k\geq 0}$ are equal. Let $\mathcal{F}$ be the σ-field generated by $\left(\tilde{X}^d_k\right)_{k\geq 0}$.

(a) $\#\{m_1,\ldots,m_4\} = 4$, as the $\left\{\left(U_{m_j}, Z^d_{m_j,1},\ldots,Z^d_{m_j,d}\right)\right\}_{1\leq j\leq 4}$ are independent conditionally to $\mathcal{F}$,

$$\begin{aligned}
\mathbb{E}\left[\prod_{j=1}^4 Z^d_{m_j,1}\mathbb{1}_{\left(\mathcal{A}^d_{m_j}\right)^c}\middle|\mathcal{F}\right] &= \prod_{j=1}^4 \mathbb{E}\left[Z^d_{m_j,1}\mathbb{1}_{\left(\tilde{\mathcal{A}}^d_{m_j}\right)^c}\middle|\mathcal{F}\right] ,\\
&= \prod_{j=1}^4 \mathbb{E}\left[Z^d_{m_j,1}\varphi\left(\sum_{i=1}^d \Delta\tilde{V}^d_{m_j-1,i}\right)\middle|\mathcal{F}\right] .
\end{aligned}$$

where $\varphi(x) = (1-\mathrm{e}^x)_+$. Since the function φ is 1-Lipschitz, we get

$$\begin{aligned}
&\left|\varphi\left(\sum_{i=1}^d \Delta\tilde{V}^d_{m_j-1,i}\right) - \varphi\left(-\frac{\ell}{\sqrt{d}}\dot{V}(\tilde{X}^d_{m_j-1,1})Z^d_{m_j,1} + \sum_{i=2}^d \Delta\tilde{V}^d_{m_j-1,i}\right)\right| \\
&\qquad\qquad \leq \left|\Delta\tilde{V}^d_{m_j-1,1} + \frac{\ell}{\sqrt{d}}\dot{V}(\tilde{X}^d_{m_j-1,1})Z^d_{m_j,1}\right| .
\end{aligned}$$

Then,

$$\left|\mathbb{E}\left[\prod_{j=1}^{4} Z^d_{m_j,1}\mathbb{1}_{\left(\mathcal{A}^d_{m_j}\right)^c}\right]\right| \le \mathbb{E}\left[\prod_{j=1}^{4}\left\{A^d_{m_j}+B^d_{m_j}\right\}\right] ,$$

where

$$\begin{aligned} A^d_{m_j} &= \mathbb{E}\left[\left|Z^d_{m_j,1}\right|\left|\Delta\tilde{V}^d_{m_j-1,1}+\frac{\ell}{\sqrt{d}}\dot{V}(\tilde{X}^d_{m_j-1,1})Z^d_{m_j,1}\right|\middle|\mathcal{F}\right] ,\\ B^d_{m_j} &= \left|\mathbb{E}\left[Z^d_{m_j,1}\left(1-\exp\left\{-\frac{\ell}{\sqrt{d}}\dot{V}(\tilde{X}^d_{m_j-1,1})Z^d_{m_j,1}+\sum_{i=2}^{d}\Delta\tilde{V}^d_{m_j-1,i}\right\}\right)_+\middle|\mathcal{F}\right]\right| . \end{aligned}$$

By the inequality of arithmetic and geometric means and convex inequalities,

$$\left|\mathbb{E}\left[\prod_{j=1}^{4} Z^d_{m_j,1}\mathbb{1}_{\left(\mathcal{A}^d_{m_j}\right)^c}\right]\right| \le 8\mathbb{E}\left[\sum_{j=1}^{4}\left(A^d_{m_j}\right)^4+\left(B^d_{m_j}\right)^4\right] .$$

By Lemma 2(ii) and the Hölder inequality, there exists $C>0$ such that $\mathbb{E}\left[\left(A^d_{m_j}\right)^4\right]\le Cd^{-2}$. On the other hand, by [2, Lemma 6] since $Z^d_{m_j,1}$ is independent of $\mathcal{F}$,

$$B^d_{m_j} = \left|\mathbb{E}\left[\frac{\ell}{\sqrt{d}}\dot{V}(\tilde{X}^d_{m_j-1,1})\mathcal{G}\left(\frac{\ell^2}{d}\dot{V}(\tilde{X}^d_{m_j-1,1})^2,-2\sum_{i=2}^{d}\Delta\tilde{V}^d_{m_j-1,i}\right)\middle|\mathcal{F}\right]\right| ,$$

where the function $\mathcal{G}$ is defined in (26). By **H**1(ii) and since $\mathcal{G}$ is bounded, $\mathbb{E}[(B^d_{m_j})^4]\le Cd^{-2}$. Therefore $|\mathbb{E}[\prod_{j=1}^4 Z^d_{m_j,1}\mathbb{1}_{(\mathcal{A}^d_{m_j})^c}]|\le Cd^{-2}$, showing that

$$\sum_{(m_1,m_2,m_3,m_4)\in\mathcal{I}_4}\left|\mathbb{E}\left[\prod_{i=1}^{4} Z^d_{m_i,1}\mathbb{1}_{(\mathcal{A}^d_{m_i})^c}\right]\right| \le \frac{C}{d^2}\binom{k_2-k_1}{4} . \tag{31}$$

(b) $\#\{m_1,\ldots,m_4\}=3$, as the $\left\{\left(U_{m_j},Z^d_{m_j,1},\ldots,Z^d_{m_j,d}\right)\right\}_{1\le j\le 3}$ are independent conditionally to $\mathcal{F}$,

$$\begin{aligned} &\left|\mathbb{E}\left[\left(Z^d_{m_1,1}\right)^2\mathbb{1}_{(\mathcal{A}^d_{m_1})^c}\prod_{j=2}^{3} Z^d_{m_j,1}\mathbb{1}_{\left(\mathcal{A}^d_{m_j}\right)^c}\middle|\mathcal{F}\right]\right| \\ &\qquad\le \mathbb{E}\left[\left(Z^d_{m_1,1}\right)^2\middle|\mathcal{F}\right]\left|\prod_{j=2}^{3}\mathbb{E}\left[Z^d_{m_j,1}\mathbb{1}_{\left(\tilde{\mathcal{A}}^d_{m_j}\right)^c}\middle|\mathcal{F}\right]\right| \le \left|\prod_{j=2}^{3}\mathbb{E}\left[Z^d_{m_j,1}\mathbb{1}_{\left(\tilde{\mathcal{A}}^d_{m_j}\right)^c}\middle|\mathcal{F}\right]\right| . \end{aligned}$$

Then, following the same steps as above, and using Holder's inequality yields

$$\left|\mathbb{E}\left[\prod_{j=2}^{3} Z^d_{m_j,1}\mathbb{1}_{\left(\mathcal{A}^d_{m_j}\right)^c}\right]\right| \le C\mathbb{E}\left[\sum_{j=2}^{3}\left(A^d_{m_j}\right)^2+\left(B^d_{m_j}\right)^2\right] \le Cd^{-1}$$

and

$$\sum_{(m_1,m_2,m_3,m_4)\in\mathcal{I}_3}\left|\mathbb{E}\left[\prod_{i=1}^{4} Z^d_{m_i,1}\mathbb{1}_{(\mathcal{A}^d_{m_i})^c}\right]\right| \le \frac{C}{d}\binom{k_2-k_1}{3} \le \frac{C}{d}(k_2-k_1)^3 . \tag{32}$$

(c) If $\#\{m_1, \ldots, m_4\} = 2$ two cases have to be considered:

$$\mathbb{E}\left[\left(Z^d_{m_1,1}\right)^2 \mathbb{1}_{\left(\tilde{\mathcal{A}}^d_{m_1}\right)^c} \left(Z^d_{m_2,1}\right)^2 \mathbb{1}_{\left(\mathcal{A}^d_{m_2}\right)^c}\right] \leq \mathbb{E}\left[\left(Z^d_{m_1,1}\right)^2\right] \mathbb{E}\left[\left(Z^d_{m_2,1}\right)^2\right] \leq 1\,,$$
$$\mathbb{E}\left[\left(Z^d_{m_1,1}\right)^3 \mathbb{1}_{\left(\mathcal{A}^d_{m_1}\right)^c} Z^d_{m_2,1} \mathbb{1}_{\left(\mathcal{A}^d_{m_2}\right)^c}\right] \leq \mathbb{E}\left[\left|Z^d_{m_1,1}\right|^3\right] \mathbb{E}\left[\left|Z^d_{m_2,1}\right|\right] \leq \frac{4}{\pi}\,.$$

This yields

$$\sum_{(m_1,m_2,m_3,m_4)\in\mathcal{I}_2} \left|\mathbb{E}\left[\prod_{i=1}^4 Z^d_{m_i,1} \mathbb{1}_{\left(\mathcal{A}^d_{m_i}\right)^c}\right]\right| \leq \left(3 + 4\cdot\frac{4}{\pi}\right)(k_2-k_1)(k_2-k_1-1) \leq C(k_2-k_1)^2\,. \quad (33)$$

(d) If $\#\{m_1, \ldots, m_4\} = 1$: $\mathbb{E}\left[\left(Z^d_{m_i,1}\mathbb{1}_{\left(\mathcal{A}^d_{m_i}\right)^c}\right)^4\right] \leq \mathbb{E}\left[\left(Z^d_{m_1,1}\right)^4\right] \leq 3$, then

$$\sum_{(m_1,m_2,m_3,m_4)\in\mathcal{I}_1} \left|\mathbb{E}\left[\prod_{i=1}^4 Z^d_{m_i,1} \mathbb{1}_{\left(\mathcal{A}^d_{m_i}\right)^c}\right]\right| \leq 3(k_2-k_1)\,. \quad (34)$$

The proof is completed by combining (29) with (53), (32), (33) and (34). □

4.5 Proof of Proposition 3

We preface the proof by a preliminary lemma.

Lemma 6. *Assume that* $\boldsymbol{H}1$ *holds. Let* μ *be a limit point of the sequence of laws* $(\mu_d)_{d\geq1}$ *of* $\left\{(Y^d_{t,1})_{t\geq0},\ d\in\mathbb{N}^*\right\}$. *Then for all* $t\geq 0$, *the pushforward measure of* μ *by* W_t *is* π.

Proof. By (7),

$$\lim_{d\to+\infty} \mathbb{E}\left[\left|Y^d_{t,1} - X^d_{\lfloor dt\rfloor,1}\right|\right] = 0\,.$$

Since $(\mu_d)_{d\geq1}$ converges weakly to μ, for all bounded Lipschitz function $\psi:\mathbb{R}\to\mathbb{R}$, $\mathbb{E}^\mu[\psi(W_t)] = \lim_{d\to+\infty}\mathbb{E}[\psi(Y^d_{t,1})] = \lim_{d\to+\infty}\mathbb{E}[\psi(X^d_{\lfloor dt\rfloor,1})]$. The proof is completed upon noting that for all $d\in\mathbb{N}^*$ and all $t\geq0$, $X^d_{\lfloor dt\rfloor,1}$ is distributed according to π . □

Proof of Proposition 3. Let μ be a limit point of $(\mu_d)_{d\geq1}$. It is straightforward to show that μ is a solution to the martingale problem associated with L if for all $\phi\in C^\infty_c(\mathbb{R},\mathbb{R})$, $m\in\mathbb{N}^*$, $g:\mathbb{R}^m\to\mathbb{R}$ bounded and continuous, and $0\leq t_1\leq\cdots\leq t_m\leq s\leq t$:

$$\mathbb{E}^\mu\left[\left(\phi\left(W_t\right) - \phi\left(W_s\right) - \int_s^t \mathrm{L}\phi\left(W_u\right)\mathrm{d}u\right) g\left(W_{t_1},\ldots,W_{t_m}\right)\right] = 0\,. \quad (35)$$

Let $\phi\in C^\infty_c(\mathbb{R},\mathbb{R})$, $m\in\mathbb{N}^*$, $g:\mathbb{R}^m\to\mathbb{R}$ continuous and bounded, $0\leq t_1\leq\cdots\leq t_m\leq s\leq t$ and $\mathbf{W}_{\dot V} = \{w\in\mathbf{W} | w_u\notin\mathcal{D}_{\dot V}$ for almost every $u\in[s,t]\}$. Note first that $w\in\mathbf{W}^c_{\dot V}$ if and only if $\int_s^t \mathbb{1}_{\mathcal{D}_{\dot V}}(w_u)\mathrm{d}u > 0$. Therefore, by $\mathbf{H}2$ and Fubini's theorem:

$$\mathbb{E}^\mu\left[\int_s^t \mathbb{1}_{\mathcal{D}_{\dot V}}(W_u)\mathrm{d}u\right] = \int_s^t \mathbb{E}^\mu\left[\mathbb{1}_{\mathcal{D}_{\dot V}}(W_u)\right]\mathrm{d}u = 0\,,$$

showing that $\mu(\mathbf{W}^c_{\dot{V}}) = 0$. We now prove that on $\mathbf{W}_{\dot{V}}$,

$$\Psi_{s,t} :\ w \mapsto \left\{ \phi(w_t) - \phi(w_s) - \int_s^t \mathrm{L}\phi(w_u)\mathrm{d}u \right\} g(w_{t_1}, \ldots, w_{t_m}) \tag{36}$$

is continuous. It is clear that it is enough to show that $w \mapsto \int_s^t \mathrm{L}\phi(w_u)\mathrm{d}u$ is continuous on $\mathbf{W}_{\dot{V}}$. So let $w \in \mathbf{W}_{\dot{V}}$ and $(w^n)_{n \geq 0}$ be a sequence in $\mathbf{W}$ which converges to w in the uniform topology on compact sets. Then by $\overline{\mathbf{H}}2$, for any u such that $w_u \notin \mathcal{D}_{\dot{V}}$, $\mathrm{L}\phi(w^n_u)$ converges to $\mathrm{L}\phi(w_u)$ when n goes to infinity and $\mathrm{L}\phi$ is bounded. Therefore by Lebesgue's dominated convergence theorem, $\int_s^t \mathrm{L}\phi(w^n_u)\mathrm{d}u$ converges to $\int_s^t \mathrm{L}\phi(w_u)\mathrm{d}u$. Hence, the map defined by (36) is continuous on $\mathbf{W}_{\dot{V}}$. Since $(\mu_d)_{d \geq 1}$ converges weakly to μ, by (16):

$$\mu\left(\Psi_{s,t}\right) = \lim_{d \to +\infty} \mu^d\left(\Psi_{s,t}\right) = 0\,,$$

which is precisely (35). □

4.6 Proof of Theorem 3

By Proposition 3, it is enough to check (16) to prove that μ is a solution to the martingale problem. The core of the proof of Theorem 3 is Proposition 6, for which we need two technical lemmata.

Lemma 7. *Let X, Y and U be $\mathbb{R}$-valued random variables and $\epsilon > 0$. Assume that U is non-negative and bounded by 1. Let $g : \mathbb{R} \to \mathbb{R}$ be a bounded function on $\mathbb{R}$ such that for all $(x, y) \in (-\infty, -\epsilon]^2 \cup [\epsilon, +\infty)^2$, $|g(x) - g(y)| \leq C_g\,|x - y|$.*

(i) For all $a > 0$,

$$\begin{aligned}\mathbb{E}\left[\mathsf{U}\,|g(\mathsf{X}) - g(Y)|\right] \leq\ & C_g \mathbb{E}\left[\mathsf{U}\,|\mathsf{X} - Y|\right] \\ & + \mathrm{osc}(g)\left\{\mathbb{P}\left[|\mathsf{X}| \leq \epsilon\right] + a^{-1}\mathbb{E}\left[\mathsf{U}\,|\mathsf{X} - Y|\right] + \mathbb{P}\left[\epsilon < |\mathsf{X}| < \epsilon + a\right]\right\}\,,\end{aligned}$$

where $\mathrm{osc}(g) = \sup(g) - \inf(g)$.

(ii) If there exist $\mu \in \mathbb{R}$ and $\sigma, C_\mathsf{X} \in \mathbb{R}_+$ such that

$$\sup_{x \in \mathbb{R}} |\mathbb{P}\left[\mathsf{X} \leq x\right] - \Phi((x - \mu)/\sigma)| \leq C_\mathsf{X}\,,$$

then

$$\begin{aligned}\mathbb{E}\left[\mathsf{U}\,|g(\mathsf{X}) - g(Y)|\right] \leq\ & C_g \mathbb{E}\left[\mathsf{U}\,|\mathsf{X} - Y|\right] \\ & + 2\,\mathrm{osc}(g)\left\{C_\mathsf{X} + \sqrt{2\mathbb{E}\left[\mathsf{U}\,|\mathsf{X} - Y|\right](2\pi\sigma^2)^{-1/2}} + \epsilon(2\pi\sigma^2)^{-1/2}\right\}\,.\end{aligned}$$

Proof. (i) Consider the following decomposition

$$\begin{aligned}\mathbb{E}\left[\mathsf{U}\,|g(\mathsf{X}) - g(\mathsf{Y})|\right] = & \ \mathbb{E}\left[\mathsf{U}\,|(g(\mathsf{X}) - g(\mathsf{Y}))|\,\mathbb{1}_{\left\{(\mathsf{X},\mathsf{Y}) \in (-\infty,-\epsilon]^2\right\} \cup \left\{(\mathsf{X},\mathsf{Y}) \in [\epsilon,+\infty)^2\right\}}\right] \\ & + \mathbb{E}\left[\mathsf{U}\,|g(\mathsf{X}) - g(\mathsf{Y})|\left(\mathbb{1}_{\{\mathsf{X} \in [-\epsilon,\epsilon]\}} + \mathbb{1}_{(\{\mathsf{X}<-\epsilon\} \cap \{\mathsf{Y} \geq -\epsilon\}) \cup (\{\mathsf{X}>\epsilon\} \cap \{\mathsf{Y} \leq \epsilon\})}\right)\right]\,.\end{aligned}$$

In addition, for all $a > 0$,

$$(\{\mathsf{X} < -\epsilon\} \cap \{\mathsf{Y} \geq -\epsilon\}) \cup (\{\mathsf{X} > \epsilon\} \cap \{\mathsf{Y} \leq \epsilon\}) \\ \subset \{\epsilon < |\mathsf{X}| < \epsilon + a\} \cup (\{|\mathsf{X}| \geq \epsilon + a\} \cap \{|\mathsf{X} - \mathsf{Y}| \geq a\}) .$$

Then using that $\mathsf{U} \in [0, 1)$, we get

$$\mathbb{E}[\mathsf{U}\,|g(\mathsf{X}) - g(\mathsf{Y})|] \leq C_g \mathbb{E}[\mathsf{U}\,|\mathsf{X} - \mathsf{Y}|] + \operatorname{osc}(g) \left(\mathbb{P}[|\mathsf{X}| < \epsilon + a] + a^{-1}\mathbb{E}[\mathsf{U}\,|\mathsf{X} - \mathsf{Y}|]\right) .$$

(ii) The result is straightforward if $\mathbb{E}[\mathsf{U}\,|\mathsf{X} - \mathsf{Y}|] = 0$. Assume $\mathbb{E}[\mathsf{U}\,|\mathsf{X} - \mathsf{Y}|] > 0$. Combining the additional assumption and the previous result,

$$\mathbb{E}[\mathsf{U}\,|g(\mathsf{X}) - g(\mathsf{Y})|] \leq C_g \mathbb{E}[\mathsf{U}\,|\mathsf{X} - \mathsf{Y}|] \\ + \operatorname{osc}(g) \left\{2C_{\mathsf{X}} + 2(\epsilon + a)(2\pi\sigma^2)^{-1/2} + a^{-1}\mathbb{E}[\mathsf{U}\,|\mathsf{X} - \mathsf{Y}|]\right\} .$$

As this result holds for all $a > 0$, the proof is concluded by setting $a = \sqrt{\mathbb{E}[\mathsf{U}\,|\mathsf{X} - \mathsf{Y}|]\,(2\pi\sigma^2)^{1/2}/2}$. □

Lemma 8. *Assume* **H1** *holds. Let X^d be distributed according to π^d and Z^d be a d-dimensional standard Gaussian random variable, independent of X^d. Then, $\lim_{d\to+\infty} \mathrm{E}^d = 0$, where*

$$\mathrm{E}^d = \mathbb{E}\left[\left|\dot{V}(X_1^d)\left\{\mathcal{G}\left(\frac{\ell^2}{d}\dot{V}(X_1^d)^2, 2\sum_{i=2}^d \Delta V_i^d\right) - \mathcal{G}\left(\frac{\ell^2}{d}\dot{V}(X_1^d)^2, 2\sum_{i=2}^d b_i^d\right)\right\}\right|\right] ,$$

ΔV_i^d and b_i^d are resp. given by (5) *and* (13)*.*

Proof. Set for all $d \geq 1$, $\bar{Y}_d = \sum_{i=2}^d \Delta V_i^d$ and $\bar{X}_d = \sum_{i=2}^d b_i^d$. By (26), $\partial_b \mathcal{G}(a, b) = -\mathcal{G}(a, b)/2 + \exp(-b^2/8a)/(2\sqrt{2\pi a})$. As $\mathcal{G}$ is bounded and $x \mapsto x\exp(-x)$ is bounded on $\mathbb{R}_+$, we get $\sup_{a\in\mathbb{R}_+;|b|\geq a^{1/4}} \partial_b\mathcal{G}(a, b) < +\infty$. Therefore, there exists $C \geq 0$ such that, for all $a \in \mathbb{R}_+$and $(b_1, b_2) \in \left(-\infty, -a^{1/4}\right)^2 \cup \left(a^{1/4}, +\infty\right)^2$,

$$|\mathcal{G}(a, b_1) - \mathcal{G}(a, b_2)| \leq C\,|b_1 - b_2| . \tag{37}$$

By definition of b_i^d (13), $\bar{X}_d$ may be expressed as $\bar{X}_d = \sigma_d \bar{S}_d + \mu_d$, where

$$\mu_d = 2(d-1)\mathbb{E}\left[\zeta^d(X_1^d, Z_1^d)\right] - \frac{\ell^2(d-1)}{4d}\mathbb{E}\left[\dot{V}(X_1^d)^2\right] ,$$
$$\sigma_d^2 = \ell^2\mathbb{E}\left[\dot{V}(X_1^d)^2\right] + \frac{\ell^4}{16d}\mathbb{E}\left[\left(\dot{V}(X_1^d)^2 - \mathbb{E}\left[\dot{V}(X_1^d)^2\right]\right)^2\right] ,$$
$$\bar{S}_d = (\sqrt{d}\sigma_d)^{-1}\sum_{i=2}^d \beta_i^d ,$$
$$\beta_i^d = -\ell Z_i^d \dot{V}(X_i^d) - \frac{\ell^2}{4\sqrt{d}}\left(\dot{V}(X_i^d)^2 - \mathbb{E}\left[\dot{V}(X_i^d)^2\right]\right) .$$

By **H**1(ii) the Berry-Essen Theorem [5, Theorem 5.7] can be applied to $\bar{S}_d$. Then, there exists a universal constant C such that for all $d > 0$,

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left[\left(\frac{d}{d-1}\right)^{1/2}\bar{S}_d \leq x\right] - \Phi(x)\right| \leq C/\sqrt{d}\,.$$

It follows that

$$\sup_{x\in\mathbb{R}} \left|\mathbb{P}\left[\bar{X}_d \leq x\right] - \Phi((x-\mu_d)/\tilde{\sigma}_d)\right| \leq C/\sqrt{d}\,,$$

where $\tilde{\sigma}_d^2 = (d-1)\sigma_d^2/d$. By this result and (37), Lemma 7 can be applied to obtain a constant $C \geq 0$, independent of d, such that:

$$\begin{aligned}\mathbb{E}\left[\left|\mathcal{G}\left(\ell^2\dot{V}(X_1^d)^2/d, 2\bar{Y}_d\right) - \mathcal{G}\left(\ell^2\dot{V}(X_1^d)^2/d, 2\bar{X}_d\right)\right| \middle| X_1^d\right] \\ \leq C\left(\varepsilon_d + d^{-1/2} + \sqrt{2\varepsilon_d(2\pi\tilde{\sigma}_d^2)^{-1/2}} + \sqrt{\ell|\dot{V}(X_1^d)|/(2\pi d^{1/2}\tilde{\sigma}_d^2)}\right)\,,\end{aligned}$$

where $\varepsilon_d = \mathbb{E}\left[\left|\bar{X}_d - \bar{Y}_d\right|\right]$. Using this result, we have

$$\begin{aligned}\mathrm{E}^d \leq C\Bigg\{\left(\varepsilon_d + d^{-1/2} + \sqrt{2\varepsilon_d(2\pi\tilde{\sigma}_d^2)^{-1/2}}\right)\mathbb{E}\left[|\dot{V}(X_1^d)|\right] \\ +\ell^{1/2}\mathbb{E}\left[|\dot{V}(X_1^d)|^{3/2}\right](2\pi d^{1/2}\tilde{\sigma}_d^2)^{-1/2}\Bigg\}\,. \quad (38)\end{aligned}$$

By Lemma 3, ε_d goes to 0 as d goes to infinity, and by $\mathbf{H}1$(ii) $\lim_{d\to+\infty}\sigma_d^2 = \ell^2\mathbb{E}\left[\dot{V}(X)^2\right]$. Combining these results with (38), it follows that E^d goes to 0 when d goes to infinity. □

For all $n \geq 0$, define $\mathcal{F}_n^d = \sigma(\{X_k^d, k \leq n\})$ and for all $\phi \in C_c^\infty(\mathbb{R},\mathbb{R})$,

$$\begin{aligned}M_n^d(\phi) = \frac{\ell}{\sqrt{d}}\sum_{k=0}^{n-1}\phi'(X_{k,1}^d)\left\{Z_{k+1,1}^d\mathbb{1}_{\mathcal{A}_{k+1}^d} - \mathbb{E}\left[Z_{k+1,1}^d\mathbb{1}_{\mathcal{A}_{k+1}^d}\,\middle|\,\mathcal{F}_k^d\right]\right\} \\ + \frac{\ell^2}{2d}\sum_{k=0}^{n-1}\phi''(X_{k,1}^d)\left\{(Z_{k+1,1}^d)^2\mathbb{1}_{\mathcal{A}_{k+1}^d} - \mathbb{E}\left[(Z_{k+1,1}^d)^2\mathbb{1}_{\mathcal{A}_{k+1}^d}\,\middle|\,\mathcal{F}_k^d\right]\right\}\,. \quad (39)\end{aligned}$$

Proposition 6. *Assume* $\boldsymbol{H}1$ *and* $\boldsymbol{H}2$ *hold. Then, for all* $s \leq t$ *and all* $\phi \in C_c^\infty(\mathbb{R},\mathbb{R})$,

$$\lim_{d\to+\infty}\mathbb{E}\left[\left|\phi(Y_{t,1}^d) - \phi(Y_{s,1}^d) - \int_s^t \mathrm{L}\phi(Y_{r,1}^d)\mathrm{d}r - \left(M_{\lceil dt\rceil}^d(\phi) - M_{\lceil ds\rceil}^d(\phi)\right)\right|\right] = 0\,.$$

Proof. First, since $\mathrm{d}Y_{r,1}^d = \ell\sqrt{d}Z_{\lceil dr\rceil,1}^d\mathbb{1}_{\mathcal{A}_{\lceil dr\rceil}^d}\mathrm{d}r$,

$$\phi(Y_{t,1}^d) - \phi(Y_{s,1}^d) = \ell\sqrt{d}\int_s^t \phi'(Y_{r,1}^d)Z_{\lceil dr\rceil,1}^d\mathbb{1}_{\mathcal{A}_{\lceil dr\rceil}^d}\mathrm{d}r\,. \quad (40)$$

As ϕ is C^3, using (7) and a Taylor expansion, for all $r \in [s,t]$ there exists $\chi_r \in \left[X_{\lfloor dr\rfloor,1}^d, Y_{r,1}^d\right]$ such that:

$$\begin{aligned}\phi'(Y_{r,1}^d) = \phi'(X_{\lfloor dr\rfloor,1}^d) + \frac{\ell}{\sqrt{d}}(dr - \lfloor dr\rfloor)\phi''(X_{\lfloor dr\rfloor,1}^d)Z_{\lceil dr\rceil,1}^d\mathbb{1}_{\mathcal{A}_{\lceil dr\rceil}^d} \\ + \frac{\ell^2}{2d}(dr-\lfloor dr\rfloor)^2\phi^{(3)}(\chi_r)\left(Z_{\lceil dr\rceil,1}^d\right)^2\mathbb{1}_{\mathcal{A}_{\lceil dr\rceil}^d}\,.\end{aligned}$$

Plugging this expression into (40) yields:

$$
\begin{aligned}
\phi(Y^d_{t,1}) - \phi(Y^d_{s,1}) = \ell\sqrt{d}\int_s^t \phi'(X^d_{\lfloor dr\rfloor,1}) Z^d_{\lceil dr\rceil,1}\mathbb{1}_{\mathcal{A}^d_{\lceil dr\rceil}}\mathrm{d}r \\
+ \ell^2\int_s^t (dr - \lfloor dr\rfloor)\phi''(X^d_{\lfloor dr\rfloor,1})(Z^d_{\lceil dr\rceil,1})^2\mathbb{1}_{\mathcal{A}^d_{\lceil dr\rceil}}\mathrm{d}r \\
+ \frac{\ell^3}{2\sqrt{d}}\int_s^t (dr - \lfloor dr\rfloor)^2\phi^{(3)}(\chi_r)(Z^d_{\lceil dr\rceil,1})^3\mathbb{1}_{\mathcal{A}^d_{\lceil dr\rceil}}\mathrm{d}r\,.
\end{aligned}
$$

As $\phi^{(3)}$ is bounded,

$$
\lim_{d\to+\infty}\mathbb{E}\left[\left|d^{-1/2}\int_s^t (dr - \lfloor dr\rfloor)^2\phi^{(3)}(\chi_r)(Z^d_{\lceil dr\rceil,1})^3\mathbb{1}_{\mathcal{A}^d_{\lceil dr\rceil}}\mathrm{d}r\right|\right] = 0\,.
$$

On the other hand, $I = \int_s^t \phi''(X^d_{\lfloor dr\rfloor,1})(dr - \lfloor dr\rfloor)(Z^d_{\lceil dr\rceil,1})^2\mathbb{1}_{\mathcal{A}^d_{\lceil dr\rceil}}\mathrm{d}r = I_1 + I_2$ with

$$
\begin{aligned}
I_1 &= \int_s^{\lceil ds\rceil/d} + \int_{\lfloor dt\rfloor/d}^t \phi''(X^d_{\lfloor dr\rfloor,1})(dr - \lfloor dr\rfloor - 1/2)(Z^d_{\lceil dr\rceil,1})^2\mathbb{1}_{\mathcal{A}^d_{\lceil dr\rceil}}\mathrm{d}r \\
I_2 &= \frac{1}{2}\int_s^t \phi''(X^d_{\lfloor dr\rfloor,1})(Z^d_{\lceil dr\rceil,1})^2\mathbb{1}_{\mathcal{A}^d_{\lceil dr\rceil}}\mathrm{d}r\,.
\end{aligned}
$$

Note that

$$
\begin{aligned}
I_1 = \frac{1}{2d}(\lceil ds\rceil - ds)(ds - \lfloor ds\rfloor)\phi''(X^d_{\lfloor ds\rfloor,1})(Z^d_{\lceil ds\rceil,1})^2\mathbb{1}_{\mathcal{A}^d_{\lceil ds\rceil}} \\
+ \frac{1}{2d}(\lceil dt\rceil - dt)(dt - \lfloor dt\rfloor)\phi''(X^d_{\lfloor dt\rfloor,1})(Z^d_{\lceil dt\rceil,1})^2\mathbb{1}_{\mathcal{A}^d_{\lceil dt\rceil}}
\end{aligned}
$$

showing, as ϕ'' is bounded, that $\lim_{d\to+\infty}\mathbb{E}[|I_1|] = 0$. Therefore,

$$
\lim_{d\to+\infty}\mathbb{E}\left[\left|\phi(Y^d_{t,1}) - \phi(Y^d_{s,1}) - I_{s,t}\right|\right] = 0\,,
$$

where

$$
I_{s,t} = \int_s^t \left\{\ell\sqrt{d}\phi'(X^d_{\lfloor dr\rfloor,1})Z^d_{\lceil dr\rceil,1} + \ell^2\phi''(X^d_{\lfloor dr\rfloor,1})(Z^d_{\lceil dr\rceil,1})^2/2\right\}\mathbb{1}_{\mathcal{A}^d_{\lceil dr\rceil}}\mathrm{d}r\,.
$$

Write

$$
I_{s,t} - \int_s^t \mathrm{L}\phi(Y^d_{r,1})\mathrm{d}r - \left(M^d_{\lceil dt\rceil}(\phi) - M^d_{\lceil ds\rceil}(\phi)\right) = T^d_1 + T^d_2 + T^d_3 - T^d_4 + T^d_5\,,
$$

where

$$
\begin{aligned}
T_1^d &= \int_s^t \phi'(X_{\lfloor dr \rfloor,1}^d)\left(\ell\sqrt{d}\,\mathbb{E}\left[Z_{\lceil dr \rceil,1}^d \mathbb{1}_{\mathcal{A}_{\lceil dr \rceil}^d}\,\middle|\,\mathcal{F}_{\lfloor dr \rfloor}^d\right] + \frac{h(\ell)}{2}\dot{V}(X_{\lfloor dr \rfloor,1}^d)\right)\mathrm{d}r\,, \\
T_2^d &= \int_s^t \phi''(X_{\lfloor dr \rfloor,1}^d)\left(\frac{\ell^2}{2}\,\mathbb{E}\left[(Z_{\lceil dr \rceil,1}^d)^2 \mathbb{1}_{\mathcal{A}_{\lceil dr \rceil}^d}\,\middle|\,\mathcal{F}_{\lfloor dr \rfloor}^d\right] - \frac{h(\ell)}{2}\right)\mathrm{d}r\,, \\
T_3^d &= \int_s^t \left(\mathrm{L}\phi(Y_{\lfloor dr \rfloor/d,1}^d) - \mathrm{L}\phi(Y_{r,1}^d)\right)\mathrm{d}r\,, \\
T_4^d &= \frac{\ell(\lceil dt \rceil - dt)}{\sqrt{d}}\phi'(X_{\lfloor dt \rfloor,1}^d)\left(Z_{\lceil dt \rceil,1}^d \mathbb{1}_{\mathcal{A}_{\lceil dt \rceil}^d} - \mathbb{E}\left[Z_{\lceil dt \rceil,1}^d \mathbb{1}_{\mathcal{A}_{\lceil dt \rceil}^d}\,\middle|\,\mathcal{F}_{\lfloor dt \rfloor}^d\right]\right) \\
&\quad + \frac{\ell^2(\lceil dt \rceil - dt)}{2d}\phi''(X_{\lfloor dt \rfloor,1}^d)\left((Z_{\lceil dt \rceil,1}^d)^2 \mathbb{1}_{\mathcal{A}_{\lceil dt \rceil}^d} - \mathbb{E}\left[(Z_{\lceil dt \rceil,1}^d)^2 \mathbb{1}_{\mathcal{A}_{\lceil dt \rceil}^d}\,\middle|\,\mathcal{F}_{\lfloor dt \rfloor}^d\right]\right)\,, \\
T_5^d &= \frac{\ell(\lceil ds \rceil - ds)}{\sqrt{d}}\phi'(X_{\lfloor ds \rfloor,1}^d)\left(Z_{\lceil ds \rceil,1}^d \mathbb{1}_{\mathcal{A}_{\lceil ds \rceil}^d} - \mathbb{E}\left[Z_{\lceil ds \rceil,1}^d \mathbb{1}_{\mathcal{A}_{\lceil ds \rceil}^d}\,\middle|\,\mathcal{F}_{\lfloor ds \rfloor}^d\right]\right) \\
&\quad + \frac{\ell^2(\lceil ds \rceil - ds)}{2d}\phi''(X_{\lfloor ds \rfloor,1}^d)\left((Z_{\lceil ds \rceil,1}^d)^2 \mathbb{1}_{\mathcal{A}_{\lceil ds \rceil}^d} - \mathbb{E}\left[(Z_{\lceil ds \rceil,1}^d)^2 \mathbb{1}_{\mathcal{A}_{\lceil ds \rceil}^d}\,\middle|\,\mathcal{F}_{\lfloor ds \rfloor}^d\right]\right)\,.
\end{aligned}
$$

It is now proved that for all $1 \leq i \leq 5$, $\lim_{d\to+\infty} \mathbb{E}[|T_i^d|] = 0$. First, as ϕ' and ϕ'' are bounded,

$$
\mathbb{E}\left[\left|T_4^d\right| + \left|T_5^d\right|\right] \leq C d^{-1/2}\,. \tag{41}
$$

Denote for all $r \in [s,t]$ and $d \geq 1$,

$$
\begin{aligned}
\Delta V_{r,i}^d &= V\left(X_{\lfloor dr \rfloor,i}^d\right) - V\left(X_{\lfloor dr \rfloor,i}^d + \ell d^{-1/2} Z_{\lceil dr \rceil,i}^d\right) \\
\Xi_r^d &= 1 \wedge \exp\left\{-\ell Z_{\lceil dr \rceil,1}^d \dot{V}(X_{\lfloor dr \rfloor,1}^d)/\sqrt{d} + \sum_{i=2}^d b_{\lfloor dr \rfloor,i}^d\right\}\,, \\
\Upsilon_r^d &= 1 \wedge \exp\left\{-\ell Z_{\lceil dr \rceil,1}^d \dot{V}(X_{\lfloor dr \rfloor,1}^d)/\sqrt{d} + \sum_{i=2}^d \Delta V_{r,i}^d\right\}\,,
\end{aligned}
$$

where for all $k, i \geq 0$, $b_{k,i}^d = b^d(X_{k,i}^d, Z_{k+1,i}^d)$, and for all $x, z \in \mathbb{R}$, $b^d(x,y)$ is given by (13). By the triangle inequality,

$$
\left|T_1^d\right| \leq \int_s^t \left|\phi'(X_{\lfloor dr \rfloor,1}^d)\right|(A_{1,r} + A_{2,r} + A_{3,r})\mathrm{d}r\,, \tag{42}
$$

where

$$
\begin{aligned}
A_{1,r} &= \left|\ell\sqrt{d}\,\mathbb{E}\left[Z_{\lceil dr \rceil,1}^d\left(\mathbb{1}_{\mathcal{A}_{\lceil dr \rceil}^d} - \Upsilon_r^d\right)\middle|\mathcal{F}_{\lfloor dr \rfloor}^d\right]\right|\,, \\
A_{2,r} &= \left|\ell\sqrt{d}\,\mathbb{E}\left[Z_{\lceil dr \rceil,1}^d\left(\Upsilon_r^d - \Xi_r^d\right)\middle|\mathcal{F}_{\lfloor dr \rfloor}^d\right]\right|\,, \\
A_{3,r} &= \left|\ell\sqrt{d}\,\mathbb{E}\left[Z_{\lceil dr \rceil,1}^d \Xi_r^d\middle|\mathcal{F}_{\lfloor dr \rfloor}^d\right] + \dot{V}(X_{\lfloor dr \rfloor,1}^d)h(\ell)/2\right|\,.
\end{aligned}
$$

Since $t \mapsto 1 \wedge \exp(t)$ is 1-Lipschitz, by Lemma 2(ii) $\mathbb{E}[\left|A_{1,r}^d\right|]$ goes to 0 as $d \to +\infty$ for almost all r. So by the Fubini theorem, the first term in (42) goes to 0 as $d \to +\infty$. For $A_{2,r}^d$, by [2, Lemma 6],

$$\mathbb{E}\left[\left|A^d_{2,r}\right|\right] \leq \mathbb{E}\left[\left|\ell^2\dot{V}(X^d_{\lfloor dr\rfloor,1})\left\{\mathcal{G}\left(\frac{\ell^2\dot{V}(X^d_{\lfloor dr\rfloor,1})^2}{d}, 2\sum_{i=2}^d \Delta V^d_{r,i}\right)\right.\right.\right.$$
$$\left.\left.\left. -\mathcal{G}\left(\frac{\ell^2\dot{V}(X^d_{\lfloor dr\rfloor,1})^2}{d}, 2\sum_{i=2}^d b^d_{\lfloor dr\rfloor,i}\right)\right\}\right|\right],$$

where $\mathcal{G}$ is defined in (26). By Lemma 8, this expectation goes to zero when d goes to infinity. Then by the Fubini theorem and the Lebesgue dominated convergence theorem, the second term of (42) goes 0 as $d \to +\infty$. For the last term, by [2, Lemma 6] again:

$$\ell\sqrt{d}\;\mathbb{E}\left[Z^d_{\lceil dr\rceil,1}\Xi^d_r\,\middle|\mathcal{F}^d_{\lfloor dr\rfloor}\right] = -\ell^2\dot{V}(X^d_{\lfloor dr\rfloor,1})$$
$$\times\,\mathcal{G}\left(\frac{\ell^2}{d}\sum_{i=1}^d \dot{V}(X^d_{\lfloor dr\rfloor,i})^2, \frac{\ell^2}{2d}\sum_{i=2}^d \dot{V}(X^d_{\lfloor dr\rfloor,i})^2 - 4(d-1)\mathbb{E}\left[\zeta^d(X,Z)\right]\right), \quad (43)$$

where X is distributed according to π and Z is a standard Gaussian random variable independent of X. As $\mathcal{G}$ is continuous on $\mathbb{R}_+\times\mathbb{R}\setminus\{0,0\}$ (see [2, Lemma 2]), by $\mathbf{H}1$(ii), Lemma 4 and the law of large numbers, almost surely,

$$\lim_{d\to+\infty}\ell^2\mathcal{G}\left(\frac{\ell^2}{d}\sum_{i=1}^d \dot{V}(X^d_{\lfloor dr\rfloor,i})^2, \frac{\ell^2}{2d}\sum_{i=2}^d \dot{V}(X^d_{\lfloor dr\rfloor,i})^2 - 4(d-1)\mathbb{E}\left[\zeta^d(X,Z)\right]\right)$$
$$= \ell^2\mathcal{G}\left(\ell^2\mathbb{E}[\dot{V}(X)^2], \ell^2\mathbb{E}[\dot{V}(X)^2]\right) = h(\ell)/2\,, \quad (44)$$

where $h(\ell)$ is defined in (10). Therefore by Fubini's Theorem, (43) and Lebesgue's dominated convergence theorem, the last term of (42) goes to 0 as d goes to infinity. The proof for T^d_2 follows the same lines. By the triangle inequality,

$$\left|T^d_2\right| \leq \left|\int_s^t \phi''(X^d_{\lfloor dr\rfloor,1})(\ell^2/2)\;\mathbb{E}\left[(Z^d_{\lceil dr\rceil,1})^2\left(\mathbb{1}_{\mathcal{A}^d_{\lceil dr\rceil}} - \Xi^d_r\right)\middle|\mathcal{F}^d_{\lfloor dr\rfloor}\right]\mathrm{d}r\right|$$
$$+\left|\int_s^t \phi''(X^d_{\lfloor dr\rfloor,1})\left((\ell^2/2)\;\mathbb{E}\left[(Z^d_{\lceil dr\rceil,1})^2\Xi^d_r\,\middle|\mathcal{F}^d_{\lfloor dr\rfloor}\right] - h(\ell)/2\right)\mathrm{d}r\right|. \quad (45)$$

By Fubini's Theorem, Lebesgue's dominated convergence theorem and Proposition 1, the expectation of the first term goes to zero when d goes to infinity. For the second term, by [2, Lemma 6 (A.5)],

$$(\ell^2/2)\mathbb{E}\left[(Z^d_{\lceil dr\rceil,1})^2 1\wedge\exp\left\{-\frac{\ell Z^d_{\lceil dr\rceil,1}}{\sqrt{d}}\dot{V}(X^d_{\lfloor dr\rfloor,1}) + \sum_{i=2}^d b^d_{\lfloor dr\rfloor,i}\right\}\middle|\mathcal{F}^d_{\lfloor dr\rfloor}\right]$$
$$= (B_1 + B_2 - B_3)/2\,, \quad (46)$$

where

$$
\begin{aligned}
B_1 &= \ell^2 \Gamma\left(\frac{\ell^2}{d}\sum_{i=1}^{d}\dot{V}(X^d_{\lfloor dr\rfloor,i})^2, \frac{\ell^2}{2d}\sum_{i=2}^{d}\dot{V}(X^d_{\lfloor dr\rfloor,i})^2 - 4(d-1)\mathbb{E}\left[\zeta^d(X,Z)\right]\right), \\
B_2 &= \frac{\ell^4 \dot{V}(X^d_{\lfloor dr\rfloor,1})^2}{d}\mathcal{G}\left(\frac{\ell^2}{d}\sum_{i=1}^{d}\dot{V}(X^d_{\lfloor dr\rfloor,i})^2, \frac{\ell^2}{2d}\sum_{i=2}^{d}\dot{V}(X^d_{\lfloor dr\rfloor,i})^2 - 4(d-1)\mathbb{E}\left[\zeta^d(X,Z)\right]\right), \\
B_3 &= \frac{\ell^4 \dot{V}(X^d_{\lfloor dr\rfloor,1})^2}{d}\left(2\pi\ell^2\sum_{i=1}^{d}\dot{V}(X^d_{\lfloor dr\rfloor,i})^2/d\right)^{-1/2} \\
&\quad \times \exp\left\{-\frac{\left[-(d-1)\mathbb{E}[2\zeta^d(X,Z)] + (\ell^2/(4d))\sum_{i=2}^{d}\dot{V}(X^d_{\lfloor dr\rfloor,i})^2\right]^2}{2\ell^2\sum_{i=1}^{d}\dot{V}(X^d_{\lfloor dr\rfloor,i})^2/d}\right\},
\end{aligned}
$$

where Γ is defined in (27). As Γ is continuous on $\mathbb{R}_+ \times \mathbb{R} \setminus \{0,0\}$ (see [2, Lemma 2]), by **H**1(ii), Lemma 4 and the law of large numbers, almost surely,

$$
\begin{aligned}
\lim_{d\to+\infty} \ell^2 \Gamma\left(\frac{\ell^2}{d}\sum_{i=1}^{d}\dot{V}(X^d_{\lfloor dr\rfloor,i})^2, \frac{\ell^2}{2d}\sum_{i=2}^{d}\dot{V}(X^d_{\lfloor dr\rfloor,i})^2 - 4(d-1)\mathbb{E}\left[\zeta^d(X,Z)\right]\right) \\
= \ell^2\Gamma\left(\ell^2\mathbb{E}[\dot{V}(X)^2], \ell^2\mathbb{E}[\dot{V}(X)^2]\right) = h(\ell). \quad (47)
\end{aligned}
$$

By Lemma 4, by **H**1(ii) and the law of large numbers, almost surely,

$$
\begin{aligned}
\lim_{d\to+\infty} \exp\left\{-\frac{\left[-(d-1)\mathbb{E}[2\zeta^d(X,Z)] + (\ell^2/(4d))\sum_{i=2}^{d}\dot{V}(X^d_{\lfloor dr\rfloor,i})^2\right]^2}{2\ell^2\sum_{i=1}^{d}\dot{V}(X^d_{\lfloor dr\rfloor,i})^2/d}\right\} \\
= \exp\left\{-\frac{\ell^2}{8}\mathbb{E}[\dot{V}(X)^2]\right\}.
\end{aligned}
$$

Then, as $\mathcal{G}$ is bounded on $\mathbb{R}_+ \times \mathbb{R}$,

$$
\lim_{d\to+\infty} \mathbb{E}\left[\left|\int_s^t \phi''(X^d_{\lfloor dr\rfloor,1})\,(B_2 - B_3)\,\mathrm{d}r\right|\right] = 0. \quad (48)
$$

Therefore, by Fubini's Theorem, (46), (47), (48) and Lebesgue's dominated convergence theorem, the second term of (45) goes to 0 as d goes to infinity. Write $T_3^d = (h(\ell)/2)(T^d_{3,1} - T^d_{3,2})$ where

$$
\begin{aligned}
T^d_{3,1} &= \int_s^t \left\{\phi''\left(X^d_{\lfloor dr\rfloor,1}\right) - \phi''\left(Y^d_{r,1}\right)\right\}\mathrm{d}r, \\
T^d_{3,2} &= \int_s^t \left\{\dot{V}\left(X^d_{\lfloor dr\rfloor,1}\right)\phi'\left(X^d_{\lfloor dr\rfloor,1}\right) - \dot{V}\left(Y^d_{r,1}\right)\phi'\left(Y^d_{r,1}\right)\right\}\mathrm{d}r.
\end{aligned}
$$

It is enough to show that $\mathbb{E}[|T^d_{3,1}|]$ and $\mathbb{E}[|T^d_{3,2}|]$ go to 0 when d goes to infinity to conclude the proof. By (7) and the mean value theorem, for all $r \in [s,t]$ there exists $\chi_r \in \left[X^d_{\lfloor dr\rfloor,1}, Y^d_{r,1}\right]$ such that

$$
\phi''\left(X^d_{\lfloor dr\rfloor,1}\right) - \phi''\left(Y^d_{r,1}\right) = \phi^{(3)}\left(\chi_r\right)(dr - \lfloor dr\rfloor)(\ell/\sqrt{d})Z^d_{\lceil dr\rceil,1}\mathbb{1}_{\mathcal{A}^d_{\lceil dr\rceil}}.
$$

Since $\phi^{(3)}$ is bounded, it follows that $\lim_{d\to+\infty}\mathbb{E}[|T^d_{3,1}|]=0$. On the other hand,

$$T^d_{3,2}=\int_s^t\left\{\dot V\left(X^d_{\lfloor dr\rfloor,1}\right)-\dot V\left(Y^d_{r,1}\right)\right\}\phi'\left(X^d_{\lfloor dr\rfloor,1}\right)\mathrm{d}r \\ +\int_s^t\left\{\phi'\left(X^d_{\lfloor dr\rfloor,1}\right)-\phi'\left(Y^d_{r,1}\right)\right\}\dot V\left(Y^d_{r,1}\right)\mathrm{d}r\,.$$

Since ϕ' has a bounded support, by **H**2, Fubini's theorem, and Lebesgue's dominated convergence theorem, the expectation of the absolute value of the first term goes to 0 as d goes to infinity. The second term is dealt with following the same steps as for $T^d_{3,1}$ and using **H**1(ii). □

Proof of Theorem 3. By Proposition 2, Proposition 3 and Proposition 6, it is enough to prove that for all $\phi\in C_c^\infty(\mathbb{R},\mathbb{R})$, $p\geq 1$, all $0\leq t_1\leq\cdots\leq t_p\leq s\leq t$ and $g:\mathbb{R}^p\to\mathbb{R}$ bounded and continuous function,

$$\lim_{d\to+\infty}\mathbb{E}\left[(M^d_{\lceil dt\rceil}(\phi)-M^d_{\lceil ds\rceil}(\phi))g(Y^d_{t_1},\ldots,Y^d_{t_p})\right]=0\,,$$

where for $n\geq 1$, $M^d_n(\phi)$ is defined in (39). But this result is straightforward taking successively the conditional expectations with respect to $\mathcal{F}_k$, for $k=\lceil dt\rceil,\ldots,\lceil ds\rceil$. □

5 Proofs of Section 3

5.1 Proof of Theorem 4

The proof of this theorem follows the same steps as the the proof of Theorem 2. Note that ξ_θ and ξ_0, given by (11), are well defined on $\mathcal{I}\cap\{x\in\mathbb{R}\mid x+\mathrm{r}\theta\in\mathcal{I}\}$. Let the function $\upsilon:\mathbb{R}^2\to\mathbb{R}$ be defined for $x,\theta\in\mathbb{R}$ by

$$\upsilon(x,\theta)=\mathbb{1}_{\mathcal{I}}(x+\mathrm{r}\theta)\mathbb{1}_{\mathcal{I}}(x+(1-\mathrm{r})\theta)\,. \tag{49}$$

Lemma 9. *Assume **G**1 holds. Then, there exists $C>0$ such that for all $\theta\in\mathbb{R}$,*

$$\left(\int_{\mathcal{I}}\left(\{\xi_\theta(x)-\xi_0(x)\}\upsilon(x,\theta)+\theta\dot V(x)\xi_0(x)/2\right)^2\mathrm{d}x\right)^{1/2}\leq C|\theta|^\beta\,.$$

Proof. The proof follows as Lemma 1 and is omitted. □

Lemma 10. *Assume that **G**1 holds. Let X be a random variable distributed according to π and Z be a standard Gaussian random variable independent of X. Define*

$$\mathcal{D}_{\mathcal{I}}=\{X+\mathrm{r}\ell d^{-1/2}Z\in\mathcal{I}\}\cap\{X+(1-\mathrm{r})\ell d^{-1/2}Z\in\mathcal{I}\}\,.$$

Then,

(i) $\lim_{d\to+\infty}d\,\left\|\mathbb{1}_{\mathcal{D}_{\mathcal{I}}}\zeta^d(X,Z)+\ell Z\dot V(X)/(2\sqrt{d})\right\|_2^2=0.$

*(ii) Let p be given by **G**1(i). Then,*

$$\lim_{d\to+\infty}\sqrt{d}\left\|\mathbb{1}_{\mathcal{D}_{\mathcal{I}}}\left\{V(X)-V(X+\ell Z/\sqrt{d})\right\}+\ell Z\dot V(X)/\sqrt{d}\right\|_p=0\,.$$

(iii) $\lim_{d\to\infty} d \left\| \mathbb{1}_{\mathcal{D}_\mathcal{I}} \left(\log(1+\zeta_d(X,Z)) - \zeta^d(X,Z) + [\zeta^d]^2(X,Z)/2 \right) \right\|_1 = 0$,

where ζ^d *is given by* (14).

Proof. Note by definition of ζ^d and ξ_θ (11), for $x \in \mathcal{I}$ and $x + \mathrm{r}\ell d^{-1/2} z \in \mathcal{I}$,

$$\zeta^d(x,z) = \xi_{\ell z d^{-1/2}}(x)/\xi_0(x) - 1 \,. \tag{50}$$

Using Lemma 9,

$$\begin{aligned}
&\left\| \mathbb{1}_{\mathcal{D}_\mathcal{I}} \zeta^d(X,Z) + \ell Z \dot{V}(X)/(2\sqrt{d}) \right\|_2^2 \\
&\qquad = \mathbb{E}\left[\int_\mathcal{I} \left(\upsilon(x, \ell Z d^{-1/2}) \left\{ \xi_{\ell Z d^{-1/2}}(x) - \xi_0(x) \right\} + \ell Z \dot{V}(x) \xi_0(x)/(2\sqrt{d}) \right)^2 \mathrm{d}x \right] \\
&\qquad \leq C \ell^{2\beta} d^{-\beta} \mathbb{E}\left[|Z|^{2\beta} \right] \,.
\end{aligned}$$

The proof of (i) is completed using $\beta > 1$. For (ii), write for all $x \in \mathcal{I}$ and $x + \ell z d^{-1/2} z \in \mathcal{I}$, $\Delta V(x,z) = V(x) - V(x + \ell z d^{-1/2})$. By **G**1(i)

$$\begin{aligned}
\left\| \mathbb{1}_{\mathcal{D}_\mathcal{I}} \Delta V(X,Z) + \ell Z \dot{V}(X)/\sqrt{d} \right\|_p^p &= \mathbb{E}\left[\int_\mathcal{I} \left(\upsilon(x, \ell Z d^{-1/2}) \Delta V(X,Z) + \ell Z \dot{V}(x)/\sqrt{d} \right)^p \pi(x) \mathrm{d}x \right] \\
&\leq C \ell^{\beta p} d^{-\beta p/2} \mathbb{E}\left[|Z|^{\beta p} \right]
\end{aligned}$$

and the proof of (ii) follows from $\beta > 1$. For (iii), note that for all $x > 0$, $u \in [0,x]$, $|(x-u)(1+u)^{-1}| \leq |x|$, and the same inequality holds for $x \in (-1,0]$ and $u \in [x,0]$. Then by (23) and (24), for all $x > -1$,

$$\left| \log(1+x) - x + x^2/2 \right| = |R(x)| \leq x^2 \left| \log(1+x) \right| \,.$$

Then by (50), for $x \in \mathcal{I}$ and $x + \ell d^{-1/2} z \in \mathcal{I}$,

$$\begin{aligned}
&\left| \log(1+\zeta_d(x,z)) - \zeta^d(x,z) + [\zeta^d]^2(x,z)/2 \right| \\
&\qquad \leq \left(\xi_{\ell z d^{-1/2}}(x)/\xi_0(x) - 1 \right)^2 \left| \log(\xi_{\ell z d^{-1/2}}(x)/\xi_0(x)) \right| \,, \\
&\qquad \leq \left(\xi_{\ell z d^{-1/2}}(x)/\xi_0(x) - 1 \right)^2 \left| V(x + \ell z d^{-1/2}) - V(x) \right| /2 \,.
\end{aligned}$$

Since for all $x \in \mathbb{R}$, $|\exp(x) - 1| \leq |x|(\exp(x)+1)$, this yields,

$$\begin{aligned}
&\left| \log(1+\zeta_d(x,z)) - \zeta^d(x,z) + [\zeta^d]^2(x,z)/2 \right| \\
&\qquad \leq \left| V(x + \ell z d^{-1/2}) - V(x) \right|^3 \left(\exp\left(V(x) - V(x + \ell z d^{-1/2}) \right) + 1 \right) /4 \,.
\end{aligned}$$

Therefore,

$$\int_\mathcal{I} \upsilon(x, \ell z d^{-1/2}) \left| \log(1+\zeta_d(x,z)) - \zeta^d(x,z) + [\zeta^d]^2(x,z)/2 \right| \pi(x) \mathrm{d}x \leq (I_1 + I_2)/4 \,,$$

where

$$\begin{aligned}
I_1 &= \int_\mathcal{I} \upsilon(x, \ell z d^{-1/2}) \left| V(x + \ell z d^{-1/2}) - V(x) \right|^3 \pi(x) \mathrm{d}x \\
I_2 &= \int_\mathcal{I} \upsilon(x, \ell z d^{-1/2}) \left| V(x + \ell z d^{-1/2}) - V(x) \right|^3 \pi(x + \ell z d^{-1/2}) \mathrm{d}x \,.
\end{aligned}$$

By Hölder's inequality, a change of variable and using **G**1(i),

$$I_1 + I_2 \le C \left(\left| \ell z d^{-1/2} \right|^3 \left(\int_{\mathcal{I}} \left| \dot{V}(x) \right|^4 \pi(x) \mathrm{d}x \right)^{3/4} + \left| \ell z d^{-1/2} \right|^{3\beta} \right) .$$

The proof follows from **G**1(ii) and $\beta > 1$.

□

For ease of notation, write for all $d \ge 1$ and $i, j \in \{1, \ldots, d\}$,

$$\mathcal{D}^d_{\mathcal{I},j} = \left\{ X^d_j + \mathrm{r}\ell d^{-1/2} Z^d_j \in \mathcal{I} \right\} \cap \left\{ X^d_j + (1-\mathrm{r})\ell d^{-1/2} Z^d_j \in \mathcal{I} \right\} ,$$

$$\mathcal{D}^d_{\mathcal{I},i:j} = \bigcap_{k=i}^{j} \mathcal{D}^d_{\mathcal{I},k} . \tag{51}$$

Lemma 11. *Assume that **G**1 holds. For all $d \ge 1$, let X^d be distributed according to π^d, and Z^d be d-dimensional Gaussian random variable independent of X^d. Then,* $\lim_{d\to+\infty} \mathrm{J}^d_{\mathcal{I}} = 0$ *where*

$$\mathrm{J}^d_{\mathcal{I}} = \left\| \mathbb{1}_{\mathcal{D}^d_{\mathcal{I},2:d}} \sum_{i=2}^{d} \left\{ \left(\Delta V^d_i + \frac{\ell Z^d_i}{\sqrt{d}} \dot{V}(X^d_i) \right) - 2\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},i}} \zeta^d(X^d_i, Z^d_i) \right] + \frac{\ell^2}{4d} \dot{V}^2(X^d_i) \right\} \right\|_1 .$$

Proof. The proof follows the same lines as the proof of Lemma 3 and is omitted. □

Define for all $d \ge 1$,

$$\mathrm{E}^d_{\mathcal{I}} = \mathbb{E}\left[\left(Z^d_1\right)^2 \left| \mathbb{1}_{\mathcal{D}^d_{\mathcal{I},1:d}} 1 \wedge \exp\left\{ \sum_{i=1}^{d} \Delta V^d_i \right\} \right.\right.$$
$$\left.\left. - 1 \wedge \exp\left\{ -\ell d^{-1/2} Z^d_1 \dot{V}(X^d_1) + \sum_{i=2}^{d} b^d_{\mathcal{I}}(X^d_i, Z^d_i) \right\} \right| \right] ,$$

where ΔV^d_i is given by (5), for all $x \in \mathcal{I}$, $z \in \mathbb{R}$,

$$b^d_{\mathcal{I}}(x,z) = -\frac{\ell z}{\sqrt{d}} \dot{V}(x) + 2\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},1}} \zeta^d(X^d_1, Z^d_1) \right] - \frac{\ell^2}{4d} \dot{V}^2(x) , \tag{52}$$

and ζ^d is given by (14).

Proposition 7. *Assume **G**1 holds. Let X^d be a random variable distributed according to π^d and Z^d be a zero-mean standard Gaussian random variable, independent of X. Then* $\lim_{d\to+\infty} \mathrm{E}^d_{\mathcal{I}} = 0$.

Proof. Let $\Lambda^d = -\ell d^{-1/2} Z^d_1 \dot{V}(X^d_1) + \sum_{i=2}^{d} \Delta V^d_i$. By the triangle inequality, $\mathrm{E}^d \le \mathrm{E}^d_1 + \mathrm{E}^d_2 + \mathrm{E}^d_3$ where

$$\mathrm{E}^d_{1,\mathcal{I}} = \mathbb{E}\left[\left(Z^d_1\right)^2 \mathbb{1}_{\mathcal{D}^d_{\mathcal{I},1:d}} \left| 1 \wedge \exp\left\{ \sum_{i=1}^{d} \Delta V^d_i \right\} - 1 \wedge \exp\left\{ \Lambda^d \right\} \right| \right] ,$$

$$\mathrm{E}^d_{2,\mathcal{I}} = \mathbb{E}\left[\left(Z^d_1\right)^2 \mathbb{1}_{\mathcal{D}^d_{\mathcal{I},2:d}} \left| 1 \wedge \exp\left\{ \Lambda^d \right\} - 1 \wedge \exp\left\{ -\ell d^{-1/2} Z^d_1 \dot{V}(X^d_1) + \sum_{i=2}^{d} b^d(X^d_i, Z^d_i) \right\} \right| \right] ,$$

$$\mathrm{E}^d_{3,\mathcal{I}} = \mathbb{E}\left[\left(Z^d_1\right)^2 \mathbb{1}_{\left(\mathcal{D}^d_{\mathcal{I},2:d}\right)^c} 1 \wedge \exp\left\{ -\ell d^{-1/2} Z^d_1 \dot{V}(X^d_1) + \sum_{i=2}^{d} b^d(X^d_i, Z^d_i) \right\} \right] ,$$

Since $t \mapsto 1 \wedge \mathrm{e}^t$ is 1-Lipschitz, by the Cauchy-Schwarz inequality we get

$$\mathrm{E}^d_{1,\mathcal{I}} \leq \mathbb{E}\left[\left(Z_1^d\right)^2 \mathbb{1}_{\mathcal{D}^d_{\mathcal{I},1}} \left|\Delta V_1^d + \ell d^{-1/2} Z_1^d \dot{V}(X_1^d)\right|\right] \leq \|Z_1^d\|_4^2 \left\|\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},1}} \Delta V_1^d + \ell d^{-1/2} Z_1^d \dot{V}(X_1^d)\right\|_2 .$$

By Lemma 2(ii), $\mathrm{E}^d_{1,\mathcal{I}}$ goes to 0 as d goes to $+\infty$. Using again that $t \mapsto 1 \wedge \mathrm{e}^t$ is 1-Lipschitz and Lemma 11, $\mathrm{E}^d_{2,\mathcal{I}}$ goes to 0 as well. Note that, as Z_1^d and $\mathbb{1}_{\left(\mathcal{D}^d_{\mathcal{I},2:d}\right)^c}$ are independent, by (18),

$$\mathrm{E}^d_{3,\mathcal{I}} \leq d\mathbb{P}\left(\left\{\mathcal{D}^d_{\mathcal{I},1}\right\}^c\right) \leq C d^{1-\gamma/2} .$$

Therefore, $\mathrm{E}^d_{3,\mathcal{I}}$ goes to 0 as d goes to $+\infty$ by **G**1(iii). □

Lemma 12. *Assume **G**1 holds. For all $d \in \mathbb{N}^*$, let X^d be a random variable distributed according to π^d and Z^d be a standard Gaussian random variable in $\mathbb{R}^d$, independent of X. Then,*

$$\lim_{d\to+\infty} 2d\,\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},1}} \zeta^d(X_1^d, Z_1^d)\right] = -\frac{\ell^2}{4} I ,$$

where I is defined in (6) *and ζ^d in* (14).

Proof. Noting that for all $\theta \in \mathbb{R}$,

$$\int_{\mathcal{I}} \mathbb{1}_{\mathcal{I}}(x + \mathrm{r}\theta)\mathbb{1}_{\mathcal{I}}(x + (1-\mathrm{r})\theta)\pi(x+\theta)\mathrm{d}x = \int_{\mathcal{I}} \mathbb{1}_{\mathcal{I}}(x + (\mathrm{r}-1)\theta)\mathbb{1}_{\mathcal{I}}(x - \mathrm{r}\theta)\pi(x)\mathrm{d}x .$$

the proof follows the same steps as the the proof of Lemma 4 and is omitted. □

Proof of Theorem 4. The proof follows the same lines as the proof of Theorem 2 and is therefore omitted. □

5.2 Proof of Proposition 4

As for the proof of Proposition 2, the proof follows from Lemma 13.

Lemma 13. *Assume **G**1. Then, there exists $C > 0$ such that, for all $0 \leq k_1 < k_2$,*

$$\mathbb{E}\left[\left(X^d_{k_2,1} - X^d_{k_1,1}\right)^4\right] \leq C \sum_{p=2}^4 \frac{(k_2-k_1)^p}{d^p} .$$

Proof. We use the same decomposition of $\mathbb{E}[(X^d_{k_2,1} - X^d_{k_1,1})^4]$ as in the proof of Lemma 5 so that we only need to upper bound the following term:

$$d^{-2}\mathbb{E}\left[\left(\sum_{k=k_1+1}^{k_2} Z^d_{k,1}\mathbb{1}_{\left(\mathcal{A}^d_k\right)^c}\right)^4\right] = d^{-2}\sum \mathbb{E}\left[\prod_{i=1}^4 Z^d_{m_i,1}\mathbb{1}_{\left(\mathcal{A}^d_{m_i}\right)^c}\right] ,$$

where the sum is over all the quadruplets $(m_p)_{p=1}^4$ satisfying $m_p \in \{k_1+1,\ldots,k_2\}$, $p = 1,\ldots,4$. Let $(m_1,m_2,m_3,m_4) \in \{k_1+1,\ldots,k_2\}^4$ and $(\tilde{X}^d_k)_{k\geq 0}$ be defined as:

$$\tilde{X}^d_0 = X^d_0 \quad \text{and} \quad \tilde{X}^d_{k+1} = \tilde{X}^d_k + \mathbb{1}_{k\notin\{m_1-1,m_2-1,m_3-1,m_4-1\}}\ell d^{-1/2} Z^d_{k+1}\mathbb{1}_{\tilde{\mathcal{A}}^d_{k+1}} ,$$

where for all $k \geq 0$ and all $1 \leq i \leq d$,

$$\begin{aligned}
\tilde{\mathcal{A}}^d_{k+1} &= \left\{ U_{k+1} \leq \exp\left(\sum_{i=1}^d \Delta \tilde{V}^d_{k,i} \right) \right\} \\
\Delta \tilde{V}^d_{k,i} &= V\left(\tilde{X}^d_{k,i}\right) - V\left(\tilde{X}^d_{k,i} + \ell d^{-1/2} Z^d_{k+1,i}\right) .
\end{aligned}$$

Define, for all $k_1 + 1 \leq k \leq k_2$, $1 \leq i, j \leq d$,

$$\begin{aligned}
\tilde{\mathcal{D}}^{d,k}_{\mathcal{I},j} &= \left\{ \tilde{X}^d_{k,j} + \mathrm{r}\ell d^{-1/2} Z^d_{k+1,j} \in \mathcal{I} \right\} \cap \left\{ \tilde{X}^d_{k,j} + (1-\mathrm{r})\ell d^{-1/2} Z^d_{k+1,j} \in \mathcal{I} \right\} , \\
\tilde{\mathcal{D}}^{d,k}_{\mathcal{I},i:j} &= \bigcap_{\ell=i}^{j} \tilde{\mathcal{D}}^{d,k}_{\mathcal{I},\ell} .
\end{aligned}$$

Note that by convention $V(x) = -\infty$ for all $x \notin \mathcal{I}$, $\tilde{\mathcal{A}}^d_{k+1} \subset \tilde{\mathcal{D}}^{d,k}_{\mathcal{I},1:d}$ so that $\left(\tilde{\mathcal{A}}^d_{k+1}\right)^c$ may be written $\left(\tilde{\mathcal{A}}^d_{k+1}\right)^c = \left(\tilde{\mathcal{D}}^{d,k}_{\mathcal{I},1:d}\right)^c \bigcup \left(\left(\tilde{\mathcal{A}}^d_{k+1}\right)^c \cap \tilde{\mathcal{D}}^{d,k}_{\mathcal{I},1:d}\right)$. Let $\mathcal{F}$ be the σ-field generated by $\left(\tilde{X}^d_k\right)_{k\geq 0}$. Consider the case $\#\{m_1, \ldots, m_4\} = 4$. The case $\#\{m_1, \ldots, m_4\} = 3$ is dealt with similarly and the two other cases follow the same lines as the proof of Lemma 13. As $\left\{\left(U_{m_j}, Z^d_{m_j,1}, \cdots, Z^d_{m_j,d}\right)\right\}_{1\leq j\leq 4}$ are independent conditionally to $\mathcal{F}$,

$$\mathbb{E}\left[\prod_{j=1}^4 Z^d_{m_j,1} \mathbb{1}_{\left(\mathcal{A}^d_{m_j}\right)^c} \middle| \mathcal{F} \right] = \prod_{j=1}^4 \left\{ \mathbb{E}\left[\mathbb{1}_{\left(\tilde{\mathcal{D}}^{d,m_j-1}_{\mathcal{I},1:d}\right)^c} Z^d_{m_j,1} \middle| \mathcal{F} \right] + \mathbb{E}\left[\mathbb{1}_{\tilde{\mathcal{D}}^{d,m_j-1}_{\mathcal{I},1:d}} \mathbb{1}_{\left(\tilde{\mathcal{A}}^d_{m_j}\right)^c} Z^d_{m_j,1} \middle| \mathcal{F} \right] \right\} .$$

As U_{m_j} is independent of $(Z^d_{m_j,1}, \cdots, Z^d_{m_j,d})$ conditionally to $\mathcal{F}$, the second term may be written:

$$\mathbb{E}\left[\mathbb{1}_{\tilde{\mathcal{D}}^{d,m_j-1}_{\mathcal{I},1:d}} \mathbb{1}_{\left(\tilde{\mathcal{A}}^d_{m_j}\right)^c} Z^d_{m_j,1} \middle| \mathcal{F} \right] = \mathbb{E}\left[\mathbb{1}_{\tilde{\mathcal{D}}^{d,m_j-1}_{\mathcal{I},1:d}} Z^d_{m_j,1} \left(1 - \exp\left\{ \sum_{i=1}^d \Delta \tilde{V}^d_{m_j-1,i} \right\}\right)_+ \middle| \mathcal{F} \right] .$$

Since the function $x \mapsto (1 - \mathrm{e}^x)_+$ is 1-Lipschitz, on $\tilde{\mathcal{D}}^{d,m_j-1}_{\mathcal{I},1:d}$

$$\left| \left(1 - \exp\left\{ \sum_{i=1}^d \Delta \tilde{V}^d_{m_j-1,i} \right\}\right)_+ - \Theta_{m_j} \right| \leq \left| \Delta \tilde{V}^d_{m_j-1,1} + \ell d^{-1/2} \dot{V}(\tilde{X}^d_{m_j-1,1}) Z^d_{m_j,1} \right| ,$$

where $\Theta_{m_j} = (1 - \exp\{-\ell d^{-1/2} \dot{V}(\tilde{X}^d_{m_j-1,1}) Z^d_{m_j,1} + \sum_{i=2}^d \Delta \tilde{V}^d_{m_j-1,i}\})_+$. Then,

$$\left| \mathbb{E}\left[\mathbb{1}_{\tilde{\mathcal{D}}^{d,m_j-1}_{\mathcal{I},1:d}} Z^d_{m_j,1} \left(1 - \exp\left\{ \sum_{i=1}^d \Delta \tilde{V}^d_{m_j-1,i} \right\}\right)_+ \middle| \mathcal{F} \right] \right| \leq A^d_{m_j} + B^d_{m_j} ,$$

where

$$\begin{aligned}
A^d_{m_j} &= \mathbb{E}\left[\left| Z^d_{m_j,1} \right| \left| \mathbb{1}_{\tilde{\mathcal{D}}^{d,m_j-1}_{\mathcal{I},1}} \Delta \tilde{V}^d_{m_j-1,1} + \ell d^{-1/2} \dot{V}(\tilde{X}^d_{m_j-1,1}) Z^d_{m_j,1} \right| \middle| \mathcal{F} \right] , \\
B^d_{m_j} &= \left| \mathbb{E}\left[\mathbb{1}_{\tilde{\mathcal{D}}^{d,m_j-1}_{\mathcal{I},2:d}} Z^d_{m_j,1} \Theta_{m_j} \middle| \mathcal{F} \right] \right| .
\end{aligned}$$

By Jensen inequality,

$$
\begin{aligned}
\left|\mathbb{E}\left[\prod_{j=1}^{4} Z^d_{m_j,1}\mathbb{1}_{\left(\mathcal{A}^d_{m_j}\right)^c}\right]\right| &\leq \mathbb{E}\left[\prod_{j=1}^{4}\left\{\mathbb{E}\left[\mathbb{1}_{\left(\tilde{\mathcal{D}}^{d,m_j-1}_{\mathcal{I},1:d}\right)^c}|Z^d_{m_j,1}|\middle|\mathcal{F}\right] + A^d_{m_j} + B^d_{m_j}\right\}\right] ,\\
&\leq C\mathbb{E}\left[\sum_{j=1}^{4}\mathbb{E}\left[\mathbb{1}_{\left(\tilde{\mathcal{D}}^{d,m_j-1}_{\mathcal{I},1:d}\right)^c}|Z^d_{m_j,1}|^4\middle|\mathcal{F}\right] + \left(A^d_{m_j}\right)^4 + \left(B^d_{m_j}\right)^4\right] ,
\end{aligned}
$$

By $\mathbf{G}1$(iii) and Holder's inequality applied with $\alpha = 1/(1-2/\gamma) > 1$, for all $1 \leq j \leq 4$,

$$
\begin{aligned}
\mathbb{E}\left[\mathbb{1}_{\left(\tilde{\mathcal{D}}^{d,m_j-1}_{\mathcal{I},1:d}\right)^c}|Z^d_{m_j,1}|^4\right] &\leq \mathbb{E}\left[\mathbb{1}_{\left(\tilde{\mathcal{D}}^{d,m_j-1}_{\mathcal{I},1}\right)^c}|Z^d_{m_j,1}|^4\right] + \sum_{i=2}^{d}\mathbb{E}\left[\mathbb{1}_{\left(\tilde{\mathcal{D}}^{d,m_j-1}_{\mathcal{I},i}\right)^c}\right] ,\\
&\leq \mathbb{E}\left[|Z^d_{m_j,1}|^{4\alpha/(\alpha-1)}\right]^{(\alpha-1)/\alpha} d^{-\gamma/(2\alpha)} + d^{1-\gamma/2} ,\\
&\leq Cd^{1-\gamma/2} .
\end{aligned}
$$

By Lemma 10(ii) and the Holder's inequality, there exists $C > 0$ such that $\mathbb{E}\left[\left(A^d_{m_j}\right)^4\right] \leq Cd^{-2}$. On the other hand, by [2, Lemma 6] since $Z^d_{m_j,1}$ is independent of $\mathcal{F}$,

$$
B^d_{m_j} = \left|\mathbb{E}\left[\mathbb{1}_{\tilde{\mathcal{D}}^{d,m_j-1}_{\mathcal{I},2:d}}\ell d^{-1/2}\dot{V}(\tilde{X}^d_{m_j-1,1})\mathcal{G}\left(\ell^2 d^{-1}\dot{V}(\tilde{X}^d_{m_j-1,1})^2, -2\sum_{i=2}^{d}\Delta\tilde{V}^d_{m_j-1,i}\right)\middle|\mathcal{F}\right]\right| ,
$$

where the function $\mathcal{G}$ is defined in (26). By $\mathbf{G}1$(ii) and since $\mathcal{G}$ is bounded, $\mathbb{E}[(B^d_{m_j})^4] \leq Cd^{-2}$. Since $\gamma \geq 6$ in $\mathbf{G}1$(iii), $|\mathbb{E}[\prod_{j=1}^4 Z^d_{m_j,1}\mathbb{1}_{(\mathcal{A}^d_{m_j})^c}]| \leq Cd^{-2}$, showing that

$$
\sum_{(m_1,m_2,m_3,m_4)\in\mathcal{I}_4}\left|\mathbb{E}\left[\prod_{i=1}^{4} Z^d_{m_i,1}\mathbb{1}_{(\mathcal{A}^d_{m_i})^c}\right]\right| \leq Cd^{-2}\binom{k_2-k_1}{4} . \tag{53}
$$

□

5.3 Proof of Proposition 5

Lemma 14. *Assume that* ***G****1 holds. Let μ be a limit point of the sequence of laws $(\mu_d)_{d\geq 1}$ of $\left\{(Y^d_{t,1})_{t\geq 0},\ d \in \mathbb{N}^*\right\}$. Then for all $t \geq 0$, the pushforward measure of μ by W_t is π.*

Proof. The proof is the same as in Lemma 6 and is omitted. □

We preface the proof by a lemma which provides a condition to verify that any limit point μ of $(\mu_d)_{d\geq 1}$ is a solution to the local martingale problem associated with (9).

Lemma 15. *Assume* ***G****1. Let μ be a limit point of the sequence $(\mu_d)_{d\geq 1}$. If for all $\phi \in C_c^\infty(\mathcal{I},\mathbb{R})$, the process $(\phi(W_t)-\phi(W_0)-\int_0^t \mathrm{L}\phi(W_u)\mathrm{d}u)_{t\geq 0}$ is a martingale with respect to μ and the filtration $(\mathscr{B}_t)_{t\geq 0}$, then μ solves the local martingale problem associated with (9).*

Proof. As for all $t \geq 0$ and $d \geq 1$, $Y_{t,1}^d \in \mathcal{I}$, for all $d \geq 1$ $\mu^d(C(\mathbb{R}_+, \overline{\mathcal{I}})) = 1$. Since $C(\mathbb{R}_+, \overline{\mathcal{I}})$ is closed in $\mathbf{W}$, we have by the Portmanteau theorem, $\mu(C(\mathbb{R}_+, \overline{\mathcal{I}})) = 1$. Therefore, we only need to prove that for all $\psi \in C^\infty(\overline{\mathcal{I}}, \mathbb{R})$, the process $(\psi(W_t) - \psi(W_0) - \int_0^t \mathrm{L}\psi(W_u)\mathrm{d}u)_{t\geq 0}$ is a local martingale with respect to μ and the filtration $(\mathscr{B}_t)_{t\geq 0}$. Let $\psi \in C^\infty(\overline{\mathcal{I}}, \mathbb{R})$.

Suppose first that for all $\varpi \in C_c^\infty(\overline{\mathcal{I}}, \mathbb{R})$, $(\varpi(W_t) - \varpi(W_t) - \int_0^t \mathrm{L}\varpi(W_u)\mathrm{d}u)_{t\geq 0}$ is a martingale. Then, consider the sequence of stopping time defined for $k \geq 1$ by $\tau_k = \inf\{t \geq 0 \mid |W_t| \geq k\}$ and a sequence $(\varpi_k)_{k\geq 0}$ in $C_c^\infty(\overline{\mathcal{I}}, \mathbb{R})$ satisfying:

1. for all $k \geq 1$ and all $x \in \overline{\mathcal{I}} \cap [-k, k]$, $\varpi_k(x) = \psi(x)$,
2. $\lim_{k\to+\infty} \varpi_k = \psi$ in $C^\infty(\overline{\mathcal{I}}, \mathbb{R})$.

Since for all $k \geq 1$,

$$\left(\psi(W_{t\wedge\tau_k}) - \psi(W_0) - \int_0^{t\wedge\tau_k} \mathrm{L}\psi(W_u)\mathrm{d}u\right)_{t\geq 0} = \left(\varpi_k(W_{t\wedge\tau_k}) - \varpi_k(W_0) - \int_0^{t\wedge\tau_k} \mathrm{L}\varpi_k(W_u)\mathrm{d}u\right)_{t\geq 0}$$

and the sequence $(\tau_k)_{k\geq 1}$ goes to $+\infty$ as k goes to $+\infty$ almost surely, it follows that $(\psi(W_t) - \psi(W_0) - \int_0^t \mathrm{L}\psi(W_u)\mathrm{d}u)_{t\geq 0}$ is a local martingale with respect to μ and the filtration $(\mathscr{B}_t)_{t\geq 0}$. It remains to show that for all $\varpi \in C_c^\infty(\overline{\mathcal{I}}, \mathbb{R})$, $(\varpi(W_t) - \varpi(W_0) - \int_0^t \mathrm{L}\varpi(W_u)\mathrm{d}u)_{t\geq 0}$ is a martingale under the assumption of the proposition. We only need to prove that for all $\varpi \in C_c^\infty(\overline{\mathcal{I}}, \mathbb{R})$, $0 \leq s \leq t$, $m \in \mathbb{N}^*$, $g : \mathbb{R}^m \to \mathbb{R}$ bounded and continuous, and $0 \leq t_1 \leq \cdots \leq t_m \leq s \leq t$:

$$\mathbb{E}^\mu\left[\left(\varpi(W_t) - \varpi(W_s) - \int_s^t \mathrm{L}\varpi(W_u)\,\mathrm{d}u\right) g(W_{t_1}, \ldots, W_{t_m})\right] = 0\,. \tag{54}$$

Let $(\phi_k)_{k\geq 0}$ be a sequence of functions in $C_c^\infty(\mathcal{I}, \mathbb{R})$ and converging to ϖ in $C_c^\infty(\overline{\mathcal{I}}, \mathbb{R})$. First note that for all $u \in [s, t]$, μ-almost everywhere,

$$\lim_{k\to+\infty} \phi_k(W_u) = \varpi(W_u)\,. \tag{55}$$

By Lemma 14, for all $u \in [s, t]$ the pushforward measure of μ by W_u has density π with respect to the Lebesgue measure and μ-almost everywhere, $\lim_{k\to+\infty} \mathrm{L}\phi_k(W_u) = \mathrm{L}\varpi(W_u)$. On the other hand, there exists $C \geq 0$ such that for all $k \geq 0$, $|\mathrm{L}\phi_k(W_u)| \leq C(1 + |\dot{V}(W_u)|)$. Then,

$$\begin{aligned}\mathbb{E}^\mu\left[\int_s^t \left(1 + |\dot{V}(W_u)|\right)\mathrm{d}u\right] &\leq (t - s) + \int_s^t \mathbb{E}^\mu\left[|\dot{V}(W_u)|\right]\mathrm{d}u \\ &\leq (t - s)\left(1 + \int_{\mathcal{I}} |\dot{V}(x)|\pi(x)\mathrm{d}x\right)\,.\end{aligned}$$

Therefore, μ-almost everywhere by $\mathbf{G}1$(ii) and the Lebesgue dominated convergence theorem, we get

$$\lim_{k\to+\infty} \int_s^t \mathrm{L}\phi_k(W_u)\mathrm{d}u = \int_s^t \mathrm{L}\varpi(W_u)\mathrm{d}u\,. \tag{56}$$

Therefore, (54) follows from (55) and (56), using again the Lebesgue dominated convergence theorem and $\mathbf{G}1$(ii). $\square$

Proof of Proposition 5. Let μ be a limit point of $(\mu_d)_{d\geq 1}$. By Lemma 15, we only need to prove that for all $\phi \in C_c^\infty(\mathcal{I}, \mathbb{R})$, the process $(\phi(W_t) - \phi(W_0) - \int_0^t \mathrm{L}\phi(W_u)\mathrm{d}u)_{t\geq 0}$ is a martingale with respect to μ and the filtration $(\mathscr{B}_t)_{t\geq 0}$. Then, the proof follows the same line as the proof of Proposition 3 and is omitted. $\square$

5.4 Proof of Theorem 5

Lemma 16. *Assume* **G***1 holds. Let X^d be distributed according to π^d and Z^d be a d-dimensional standard Gaussian random variable, independent of X^d. Then, $\lim_{d\to+\infty} \mathrm{E}^d = 0$, where*

$$\mathrm{E}^d = \mathbb{E}\left[\left|\dot{V}(X_1^d)\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},2:d}}\left\{\mathcal{G}\left(\ell^2\dot{V}(X_1^d)^2/d, 2\bar{Y}_d\right) - \mathcal{G}\left(\ell^2\dot{V}(X_1^d)^2/d, 2\bar{X}_d\right)\right\}\right|\right] ,$$

where $\bar{Y}_d = \sum_{i=2}^d \Delta V_i^d$, ΔV_i^d and $\mathcal{D}^d_{\mathcal{I},2:d}$ are given by (5) and (51) and $\bar{X}_d = \sum_{i=2}^d b^d_{\mathcal{I},i}$, $b^d_{\mathcal{I},i} = b^d_{\mathcal{I}}(X_i^d, Z_i^d)$ with $b^d_{\mathcal{I}}$ given by (52).

Proof. Set for all $d \geq 1$, $\bar{Y}_d = \sum_{i=2}^d \Delta V_i^d$ and $\bar{X}_d = \sum_{i=2}^d b^d_{\mathcal{I},i}$. By definition of $b^d_{\mathcal{I}}$ (52), $\bar{X}_d$ may be expressed as $\bar{X}_d = \sigma_d \bar{S}_d + \mu_d$, where

$$\begin{aligned}
\mu_d &= 2(d-1)\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},1}}\zeta^d(X_1^d, Z_1^d)\right] - \frac{\ell^2(d-1)}{4d}\mathbb{E}\left[\dot{V}(X_1^d)^2\right] ,\\
\sigma_d^2 &= \ell^2\mathbb{E}\left[\dot{V}(X_1^d)^2\right] + \frac{\ell^4}{16d}\mathbb{E}\left[\left(\dot{V}(X_1^d)^2 - \mathbb{E}\left[\dot{V}(X_1^d)^2\right]\right)^2\right] ,\\
\bar{S}_d &= (\sqrt{d}\sigma_d)^{-1}\sum_{i=2}^d \beta_i^d ,\\
\beta_i^d &= -\ell Z_i^d \dot{V}(X_i^d) - \frac{\ell^2}{4\sqrt{d}}\left(\dot{V}(X_i^d)^2 - \mathbb{E}\left[\dot{V}(X_i^d)^2\right]\right) .
\end{aligned}$$

By **G**1(ii) the Berry-Essen Theorem [5, Theorem 5.7] can be applied to $\bar{S}_d$. Then, there exists a universal constant C such that for all $d > 0$,

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left[\sqrt{\frac{d}{d-1}}\bar{S}_d \leq x\right] - \Phi(x)\right| \leq C/\sqrt{d} .$$

It follows, with $\tilde{\sigma}_d^2 = (d-1)\sigma_d^2/d$, that

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left[\bar{X}_d \leq x\right] - \Phi((x-\mu_d)/\tilde{\sigma}_d)\right| \leq C/\sqrt{d} .$$

By this result and (37), Lemma 7 can be applied to obtain a constant $C \geq 0$, independent of d, such that:

$$\begin{aligned}
&\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},2:d}}\left|\mathcal{G}\left(\ell^2\dot{V}(X_1^d)^2/d, 2\bar{Y}_d\right) - \mathcal{G}\left(\ell^2\dot{V}(X_1^d)^2/d, 2\bar{X}_d\right)\right|X_1^d\right|\right] \\
&\quad\leq C\left(\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},2:d}}\left|\bar{X}_d - \bar{Y}_d\right|\right] + d^{-1/2} + \sqrt{2\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},2:d}}\left|\bar{X}_d - \bar{Y}_d\right|\right](2\pi\tilde{\sigma}_d^2)^{-1/2}}\right.\\
&\qquad\left.+\sqrt{\ell|\dot{V}(X_1^d)|/(2\pi d^{1/2}\tilde{\sigma}_d^2)}\right) .
\end{aligned}$$

Using this result, we have

$$\mathrm{E}^d \leq C\left\{\ell^{1/2}\mathbb{E}\left[|\dot{V}(X_1^d)|^{3/2}\right](2\pi d^{1/2}\tilde{\sigma}_d^2)^{-1/2} + \mathbb{E}\left[|\dot{V}(X_1^d)|\right] \right. \\ \left. \times\left(\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},2:d}}\left|\bar{X}_d - \bar{Y}_d\right|\right] + d^{-1/2} + \sqrt{2\mathbb{E}\left[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},2:d}}\left|\bar{X}_d - \bar{Y}_d\right|\right](2\pi\tilde{\sigma}_d^2)^{-1/2}}\right)\right\} . \tag{57}$$

By Lemma 11, $\mathbb{E}[\mathbb{1}_{\mathcal{D}^d_{\mathcal{I},2:d}}|\bar{X}_d - \bar{Y}_d|]$ goes to 0 as d goes to infinity, and by **G**1(ii) $\lim_{d\to+\infty}\tilde{\sigma}_d^2 = \ell^2\mathbb{E}\left[\dot{V}(X)^2\right]$. Combining these results with (57), it follows that E^d goes to 0 when d goes to infinity. □

For all $n \geq 0$, define $\mathcal{F}_n^d = \sigma(\{X_k^d, k \leq n\})$ and for all $\phi \in C_c^\infty(\mathbb{R},\mathbb{R})$,

$$M_n^d(\phi) = \frac{\ell}{\sqrt{d}}\sum_{k=0}^{n-1}\phi'(X_{k,1}^d)\left\{Z_{k+1,1}^d\mathbb{1}_{\mathcal{A}_{k+1}^d} - \mathbb{E}\left[Z_{k+1,1}^d\mathbb{1}_{\mathcal{A}_{k+1}^d}\,\middle|\,\mathcal{F}_k^d\right]\right\} \\ + \frac{\ell^2}{2d}\sum_{k=0}^{n-1}\phi''(X_{k,1}^d)\left\{(Z_{k+1,1}^d)^2\mathbb{1}_{\mathcal{A}_{k+1}^d} - \mathbb{E}\left[(Z_{k+1,1}^d)^2\mathbb{1}_{\mathcal{A}_{k+1}^d}\,\middle|\,\mathcal{F}_k^d\right]\right\} . \tag{58}$$

Proposition 8. *Assume **G**1 and **G**2 hold. Then, for all $s \leq t$ and all $\phi \in C_c^\infty(\mathbb{R},\mathbb{R})$,*

$$\lim_{d\to+\infty}\mathbb{E}\left[\left|\phi(Y_{t,1}^d) - \phi(Y_{s,1}^d) - \int_s^t \mathrm{L}\phi(Y_{r,1}^d)\mathrm{d}r - \left(M_{\lceil dt\rceil}^d(\phi) - M_{\lceil ds\rceil}^d(\phi)\right)\right|\right] = 0 .$$

Proof. Using the same decomposition as in the proof of Proposition 6, we only need to prove that for all $1 \leq i \leq 5$, $\lim_{d\to+\infty}\mathbb{E}[|T_i^d|] = 0$, where

$$\begin{aligned}
T_1^d &= \int_s^t \phi'(X_{\lfloor dr\rfloor,1}^d)\left(\ell\sqrt{d}\,\mathbb{E}\left[Z_{\lceil dr\rceil,1}^d\mathbb{1}_{\mathcal{A}_{\lceil dr\rceil}^d}\,\middle|\,\mathcal{F}_{\lfloor dr\rfloor}^d\right] + \frac{h(\ell)}{2}\dot{V}(X_{\lfloor dr\rfloor,1}^d)\right)\mathrm{d}r ,\\
T_2^d &= \int_s^t \phi''(X_{\lfloor dr\rfloor,1}^d)\left(\frac{\ell^2}{2}\,\mathbb{E}\left[(Z_{\lceil dr\rceil,1}^d)^2\mathbb{1}_{\mathcal{A}_{\lceil dr\rceil}^d}\,\middle|\,\mathcal{F}_{\lfloor dr\rfloor}^d\right] - \frac{h(\ell)}{2}\right)\mathrm{d}r ,\\
T_3^d &= \int_s^t \left(\mathrm{L}\phi(Y_{\lfloor dr\rfloor/d,1}^d) - \mathrm{L}\phi(Y_{r,1}^d)\right)\mathrm{d}r ,\\
T_4^d &= \frac{\ell(\lceil dt\rceil - dt)}{\sqrt{d}}\phi'(X_{\lfloor dt\rfloor,1}^d)\left(Z_{\lceil dt\rceil,1}^d\mathbb{1}_{\mathcal{A}_{\lceil dt\rceil}^d} - \mathbb{E}\left[Z_{\lceil dt\rceil,1}^d\mathbb{1}_{\mathcal{A}_{\lceil dt\rceil}^d}\,\middle|\,\mathcal{F}_{\lfloor dt\rfloor}^d\right]\right)\\
&\quad + \frac{\ell^2(\lceil dt\rceil - dt)}{2d}\phi''(X_{\lfloor dt\rfloor,1}^d)\left((Z_{\lceil dt\rceil,1}^d)^2\mathbb{1}_{\mathcal{A}_{\lceil dt\rceil}^d} - \mathbb{E}\left[(Z_{\lceil dt\rceil,1}^d)^2\mathbb{1}_{\mathcal{A}_{\lceil dt\rceil}^d}\,\middle|\,\mathcal{F}_{\lfloor dt\rfloor}^d\right]\right) ,\\
T_5^d &= \frac{\ell(\lceil ds\rceil - ds)}{\sqrt{d}}\phi'(X_{\lfloor ds\rfloor,1}^d)\left(Z_{\lceil ds\rceil,1}^d\mathbb{1}_{\mathcal{A}_{\lceil ds\rceil}^d} - \mathbb{E}\left[Z_{\lceil ds\rceil,1}^d\mathbb{1}_{\mathcal{A}_{\lceil ds\rceil}^d}\,\middle|\,\mathcal{F}_{\lfloor ds\rfloor}^d\right]\right)\\
&\quad + \frac{\ell^2(\lceil ds\rceil - ds)}{2d}\phi''(X_{\lfloor ds\rfloor,1}^d)\left((Z_{\lceil ds\rceil,1}^d)^2\mathbb{1}_{\mathcal{A}_{\lceil ds\rceil}^d} - \mathbb{E}\left[(Z_{\lceil ds\rceil,1}^d)^2\mathbb{1}_{\mathcal{A}_{\lceil ds\rceil}^d}\,\middle|\,\mathcal{F}_{\lfloor ds\rfloor}^d\right]\right) .
\end{aligned}$$

First, as ϕ' and ϕ'' are bounded, $\mathbb{E}\left[\left|T_4^d\right| + \left|T_5^d\right|\right] \leq Cd^{-1/2}$. Denote for all $r \in [s,t]$ and $d \geq 1$,

$$\begin{aligned}
\Delta V_{r,i}^d &= V\left(X_{\lfloor dr\rfloor,i}^d\right) - V\left(X_{\lfloor dr\rfloor,i}^d + \ell d^{-1/2}Z_{\lceil dr\rceil,i}^d\right)\\
\Xi_r^d &= 1 \wedge \exp\left\{-\ell Z_{\lceil dr\rceil,1}^d\dot{V}(X_{\lfloor dr\rfloor,1}^d)/\sqrt{d} + \sum_{i=2}^d b_{\mathcal{I},i}^{d,\lfloor dr\rfloor}\right\} ,
\end{aligned}$$

where for all $k, i \geq 0$, $b_{\mathcal{I},i}^{d,k} = b_{\mathcal{I}}^{d}(X_{k,i}^d, Z_{k+1,i}^d)$, and for all $x, z \in \mathbb{R}$, $b_{\mathcal{I}}^d(x,y)$ is given by (52). For all $k \geq 0$, $1 \leq i, j \leq d$, define

$$\begin{aligned}
\mathcal{D}_{\mathcal{I},j}^{d,k} &= \left\{X_{k,j}^d + \mathrm{r}\ell d^{-1/2} Z_{k+1,j}^d \in \mathcal{I}\right\} \cap \left\{X_{k,j}^d + (1-\mathrm{r})\ell d^{-1/2} Z_{k+1,j}^d \in \mathcal{I}\right\} \\
\mathcal{D}_{\mathcal{I},i:j}^{d,k} &= \bigcap_{\ell=i}^{j} \mathcal{D}_{\mathcal{I},\ell}^{d,k} .
\end{aligned}$$

By the triangle inequality,

$$|T_1| \leq \int_s^t \left|\phi'(X_{\lfloor dr \rfloor,1}^d)\right| (A_{1,r} + A_{2,r} + A_{3,r} + A_{4,r}) \mathrm{d}r \,, \tag{59}$$

where

$$\begin{aligned}
\Pi_r^d &= 1 \wedge \exp\left\{-\ell d^{-1/2} Z_{\lceil dr \rceil,1}^d \dot{V}(X_{\lfloor dr \rfloor,1}^d) + \sum_{i=2}^d \Delta V_{r,i}^d\right\} \,, \\
A_{1,r} &= \left|\ell\sqrt{d}\mathbb{E}\left[Z_{\lceil dr \rceil,1}^d \left(\mathbb{1}_{\mathcal{A}_{\lceil dr \rceil}^d} - \mathbb{1}_{\mathcal{D}_{\mathcal{I},1:d}^{d,\lfloor dr \rfloor}} \Pi_r^d\right) \middle| \mathcal{F}_{\lfloor dr \rfloor}^d\right]\right| \,, \\
A_{2,r} &= \left|\ell\sqrt{d}\; \mathbb{E}\left[Z_{\lceil dr \rceil,1}^d \mathbb{1}_{\mathcal{D}_{\mathcal{I},1:d}^{d,\lfloor dr \rfloor}} \left(\Pi_r^d - \Xi_r^d\right) \middle| \mathcal{F}_{\lfloor dr \rfloor}^d\right]\right| \,, \\
A_{3,r} &= \left|\ell\sqrt{d}\; \mathbb{E}\left[Z_{\lceil dr \rceil,1}^d \mathbb{1}_{\left(\mathcal{D}_{\mathcal{I},1:d}^{d,\lfloor dr \rfloor}\right)^c} \Xi_r^d \middle| \mathcal{F}_{\lfloor dr \rfloor}^d\right]\right| \,, \\
A_{4,r} &= \left|\ell\sqrt{d}\; \mathbb{E}\left[Z_{\lceil dr \rceil,1}^d \Xi_r^d \middle| \mathcal{F}_{\lfloor dr \rfloor}^d\right] + \dot{V}(X_{\lfloor dr \rfloor,1}^d) h(\ell)/2\right| \,.
\end{aligned}$$

Since $t \mapsto 1 \wedge \exp(t)$ is 1-Lipschitz,

$$\begin{aligned}
\mathbb{E}\left[\left|A_{1,r}^d\right|\right] &\leq \ell\sqrt{d}\mathbb{E}\left[\mathbb{1}_{\mathcal{D}_{\mathcal{I},1:d}^{d,\lfloor dr \rfloor}} \left|Z_{\lceil dr \rceil,1}^d\right| \left|\Delta V_{r,1}^d - \ell d^{-1/2} Z_{\lceil dr \rceil,1}^d \dot{V}(X_{\lfloor dr \rfloor,1}^d)\right|\right] \,, \\
&\leq \ell\sqrt{d}\mathbb{E}\left[\mathbb{1}_{\mathcal{D}_{\mathcal{I},1}^{d,\lfloor dr \rfloor}} \left|Z_{\lceil dr \rceil,1}^d\right| \left|\Delta V_{r,1}^d - \ell d^{-1/2} Z_{\lceil dr \rceil,1}^d \dot{V}(X_{\lfloor dr \rfloor,1}^d)\right|\right] \,, \\
&\leq \ell\sqrt{d}\mathbb{E}\left[\left|Z_{\lceil dr \rceil,1}^d\right| \left|\mathbb{1}_{\mathcal{D}_{\mathcal{I},1}^{d,\lfloor dr \rfloor}} \Delta V_{r,1}^d - \ell d^{-1/2} Z_{\lceil dr \rceil,1}^d \dot{V}(X_{\lfloor dr \rfloor,1}^d)\right|\right]
\end{aligned}$$

and $\mathbb{E}[\left|A_{1,r}^d\right|]$ goes to 0 as $d \to +\infty$ for almost all r by Lemma 10(ii). So by the Fubini theorem, the first term in (59) goes to 0 as $d \to +\infty$. For $A_{2,r}^d$, note that

$$A_{2,r} \leq \left|\ell\sqrt{d}\; \mathbb{E}\left[Z_{\lceil dr \rceil,1}^d \mathbb{1}_{\mathcal{D}_{\mathcal{I},2:d}^{d,\lfloor dr \rfloor}} \left(\Pi_r^d - \Xi_r^d\right) \middle| \mathcal{F}_{\lfloor dr \rfloor}^d\right]\right| \,.$$

Then, by [2, Lemma 6],

$$\begin{aligned}
\mathbb{E}\left[\left|A_{2,r}^d\right|\right] \leq \mathbb{E}\Bigg[\Bigg|\ell^2 \dot{V}(X_{\lfloor dr \rfloor,1}^d) \mathbb{1}_{\mathcal{D}_{\mathcal{I},2:d}^{d,\lfloor dr \rfloor}} \Bigg\{&\mathcal{G}\left(\frac{\ell^2 \dot{V}(X_{\lfloor dr \rfloor,1}^d)^2}{d}, 2\sum_{i=2}^d \Delta V_{r,i}^d\right) \\
&-\mathcal{G}\left(\frac{\ell^2 \dot{V}(X_{\lfloor dr \rfloor,1}^d)^2}{d}, 2\sum_{i=2}^d b_{\mathcal{I},i}^{d,\lfloor dr \rfloor}\right)\Bigg\}\Bigg|\Bigg] \,,
\end{aligned}$$

where $\mathcal{G}$ is defined in (26). By Lemma 16, this expectation goes to zero when d goes to infinity. Then by the Fubini theorem and the Lebesgue dominated convergence theorem, the second term of (59) goes 0 as $d \to +\infty$. On the other hand, by **G**1(iii) and Holder's inequality applied with $\alpha = 1/(1-2/\gamma) > 1$, for all $1 \leq j \leq 4$,

$$\begin{aligned}\mathbb{E}\left[\left|A^d_{3,r}\right|\right] &\leq \ell\sqrt{d}\left(\mathbb{E}\left[\left|Z^d_{\lceil dr\rceil,1}\right|\mathbb{1}_{\left(\mathcal{D}^{d,\lfloor dr\rfloor}_{\mathcal{I},1}\right)^c}\right] + \sum_{i=2}^d \mathbb{E}\left[\mathbb{1}_{\left(\mathcal{D}^{d,\lfloor dr\rfloor}_{\mathcal{I},i}\right)^c}\right]\right)\,, \\ &\leq \ell\sqrt{d}\left(\mathbb{E}\left[|Z^d_{m_j,1}|^{\alpha/(\alpha-1)}\right]^{(\alpha-1)/\alpha} d^{-\gamma/(2\alpha)} + d^{1-\gamma/2}\right) \leq C d^{3/2-\gamma/2}\end{aligned}$$

and $\mathbb{E}[\left|A^d_{3,r}\right|]$ goes to 0 as $d \to +\infty$ for almost all r. Define

$$\bar{V}_{d,1} = \sum_{i=1}^d \dot{V}(X^d_{\lfloor dr\rfloor,i})^2 \quad \text{and} \quad \bar{V}_{d,2} = \bar{V}_{d,1} - \dot{V}(X^d_{\lfloor dr\rfloor,1})^2\,.$$

For the last term, by [2, Lemma 6]:

$$\begin{aligned}\ell\sqrt{d}\;\mathbb{E}\left[Z^d_{\lceil dr\rceil,1}\Xi^d_r\,\middle|\mathcal{F}^d_{\lfloor dr\rfloor}\right] &= -\ell^2\dot{V}(X^d_{\lfloor dr\rfloor,1}) \\ &\times \mathcal{G}\left(\frac{\ell^2}{d}\bar{V}_{d,1}, \left\{\frac{\ell^2}{2d}\bar{V}_{d,2} - 4(d-1)\mathbb{E}\left[\mathbb{1}_{\mathcal{D}_{\mathcal{I}}}\zeta^d(X,Z)\right]\right\}\right)\,, \quad (60)\end{aligned}$$

where $\mathcal{D}_{\mathcal{I}} = \left\{X + \ell d^{-1/2}Z \in \mathcal{I}\right\}$, X is distributed according to π and Z is a standard Gaussian random variable independent of X. As $\mathcal{G}$ is continuous on $\mathbb{R}_+ \times \mathbb{R} \setminus \{0,0\}$ (see [2, Lemma 2]), by **G**1(ii), Lemma 12 and the law of large numbers, almost surely,

$$\begin{aligned}\lim_{d\to+\infty} \ell^2\mathcal{G}\left(\ell^2\bar{V}_{d,1}/d, \ell^2\bar{V}_{d,2}/(2d) - 4(d-1)\mathbb{E}\left[\mathbb{1}_{\mathcal{D}_{\mathcal{I}}}\zeta^d(X,Z)\right]\right) \\ = \ell^2\mathcal{G}\left(\ell^2\mathbb{E}[\dot{V}(X)^2], \ell^2\mathbb{E}[\dot{V}(X)^2]\right) = h(\ell)/2\,, \quad (61)\end{aligned}$$

where $h(\ell)$ is defined in (10). Therefore by Fubini's Theorem, (60) and Lebesgue's dominated convergence theorem, the last term of (59) goes to 0 as d goes to infinity. The proof for T^d_2 follows the same lines. By the triangle inequality,

$$\begin{aligned}\left|T^d_2\right| \leq \left|\int_s^t \phi''(X^d_{\lfloor dr\rfloor,1})(\ell^2/2)\;\mathbb{E}\left[(Z^d_{\lceil dr\rceil,1})^2\left(\mathbb{1}_{\mathcal{A}^d_{\lceil dr\rceil}} - \Xi^d_r\right)\middle|\mathcal{F}^d_{\lfloor dr\rfloor}\right]\mathrm{d}r\right| \\ + \left|\int_s^t \phi''(X^d_{\lfloor dr\rfloor,1})\left((\ell^2/2)\;\mathbb{E}\left[(Z^d_{\lceil dr\rceil,1})^2\Xi^d_r\middle|\mathcal{F}^d_{\lfloor dr\rfloor}\right] - h(\ell)/2\right)\mathrm{d}r\right|\,. \quad (62)\end{aligned}$$

By Fubini's Theorem, Lebesgue's dominated convergence theorem and Proposition 7, the expectation of the first term goes to zero when d goes to infinity. For the second term, by [2, Lemma 6 (A.5)],

$$\begin{aligned}(\ell^2/2)\mathbb{E}\left[(Z^d_{\lceil dr\rceil,1})^2 1 \wedge \exp\left\{-\frac{\ell Z^d_{\lceil dr\rceil,1}}{\sqrt{d}}\dot{V}(X^d_{\lfloor dr\rfloor,1}) + \sum_{i=2}^d b^{d,\lfloor dr\rfloor}_{\mathcal{I},i}\right\}\middle|\mathcal{F}^d_{\lfloor dr\rfloor}\right] \\ = (B_1 + B_2 - B_3)/2\,, \quad (63)\end{aligned}$$

where

$$
\begin{aligned}
B_1 &= \ell^2 \Gamma\left(\ell^2 \bar{V}_{d,1}/d, \ell^2 \bar{V}_{d,2}/(2d) - 4(d-1)\mathbb{E}\left[\mathbb{1}_{\mathcal{D}_\mathcal{I}} \zeta^d(X,Z)\right]\right) , \\
B_2 &= \frac{\ell^4 \dot{V}(X^d_{\lfloor dr \rfloor,1})^2}{d} \mathcal{G}\left(\ell^2 \bar{V}_{d,1}/d, \ell^2 \bar{V}_{d,2}/(2d) - 4(d-1)\mathbb{E}\left[\mathbb{1}_{\mathcal{D}_\mathcal{I}} \zeta^d(X,Z)\right]\right) , \\
B_3 &= \frac{\ell^4 \dot{V}(X^d_{\lfloor dr \rfloor,1})^2}{d} \left(2\pi \ell^2 \bar{V}_{d,1}/d\right)^{-1/2} \\
&\qquad \times \exp\left\{ - \frac{\left[-2(d-1)\mathbb{E}[\mathbb{1}_{\mathcal{D}_\mathcal{I}} \zeta^d(X,Z)] + (\ell^2/(4d))\bar{V}_{d,2}\right]^2}{2\ell^2 \bar{V}_{d,1}/d} \right\} ,
\end{aligned}
$$

where Γ is defined in (27). As Γ is continuous on $\mathbb{R}_+ \times \mathbb{R} \setminus \{0,0\}$ (see [2, Lemma 2]), by **G**1(ii), Lemma 12 and the law of large numbers, almost surely,

$$
\begin{aligned}
\lim_{d\to+\infty} \ell^2 \Gamma\left(\ell^2 \bar{V}_{d,1}/d, \left\{\ell^2 \bar{V}_{d,2}/(2d) - 4(d-1)\mathbb{E}\left[\mathbb{1}_{\mathcal{D}_\mathcal{I}} \zeta^d(X,Z)\right]\right\}\right) \\
= \ell^2 \Gamma\left(\ell^2 \mathbb{E}[\dot{V}(X)^2], \ell^2 \mathbb{E}[\dot{V}(X)^2]\right) = h(\ell) . \quad (64)
\end{aligned}
$$

By Lemma 12, by **G**1(ii) and the law of large numbers, almost surely,

$$
\lim_{d\to+\infty} \exp\left\{ - \frac{\left[-2(d-1)\mathbb{E}[\mathbb{1}_{\mathcal{D}_\mathcal{I}} \zeta^d(X,Z)] + (\ell^2/(4d))\bar{V}_{d,2}\right]^2}{2\ell^2 \bar{V}_{d,1}/d} \right\} = \exp\left\{ -\frac{\ell^2}{8} \mathbb{E}[\dot{V}(X)^2] \right\} .
$$

Then, as $\mathcal{G}$ is bounded on $\mathbb{R}_+ \times \mathbb{R}$,

$$
\lim_{d\to+\infty} \mathbb{E}\left[\left| \int_s^t \phi''(X^d_{\lfloor dr \rfloor,1}) \left(B_2 - B_3\right) \mathrm{d}r \right|\right] = 0 . \quad (65)
$$

Therefore, by Fubini's Theorem, (63), (64), (65) and Lebesgue's dominated convergence theorem, the second term of (62) goes to 0 as d goes to infinity. The proof for T^d_3 follows exactly the same lines as the proof of Proposition 6. □

Proof of Theorem 5. Using Proposition 4, Proposition 5 and Proposition 8, the proof follows the same lines as the proof of Theorem 3. □

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